

# A DIRICHLET AND A THOMSON PRINCIPLE FOR NON-SELFADJOINT ELLIPTIC OPERATORS, METASTABILITY IN NON-REVERSIBLE DIFFUSION PROCESSES.

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ABSTRACT. We present two variational formulae for the capacity in the context of non-selfadjoint elliptic operators. The minimizers of these variational problems are expressed as solutions of boundary-value elliptic equations. We use these principles to provide a sharp estimate for the transition times between two different wells for non-reversible diffusion processes. This estimate permits to describe the metastable behavior of the system.

## 1. INTRODUCTION

This article is divided in two parts. In the first one, we present two variational formulae which extend to non-selfadjoint elliptic operators the classical Dirichlet and Thomson principle. In the second one, we use these formulae to describe the metastable behavior of a non-reversible diffusion process in a double-well potential field.

Fix a smooth, bounded, domain (open and connected)  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , and a smooth function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . Denote by  $\Omega_f$  the set of functions  $v : \bar{\Omega} \rightarrow \mathbb{R}$  such that  $v = f$  on  $\partial\Omega$ , the boundary of  $\Omega$ . The classical Dirichlet principle [15, 1] states that the energy

$$\int_{\Omega} \|\nabla u(x)\|^2 dx$$

is minimized on  $\Omega_f$  by the harmonic function on  $\Omega$  which takes the value  $f$  at the boundary, that is, by the solution of

$$\Delta u = 0 \text{ on } \Omega \text{ and } u = f \text{ on } \partial\Omega. \quad (1.1)$$

When  $\Omega = D \setminus \bar{B}$ , where  $B \subset D$  are smooth domains, and  $f = 1, 0$  on  $B, D^c$ , respectively, the minimal energy is called the capacity. In electrostatics, it corresponds to the total electric charge on the conductor  $\partial B$  held at unit potential and grounded at  $D^c$ . It is denoted by  $\text{cap}_D(B)$  and can be represented, by the divergence theorem, as

$$\text{cap}_D(B) = - \int_{\partial B} \frac{\partial h}{\partial \mathbf{n}_B} d\sigma, \quad (1.2)$$

where  $h$  is the harmonic function which solves (1.1),  $\mathbf{n}_B$  is the outward normal vector to  $\partial B$ , and  $\sigma$  the surface measure at  $\partial B$ .

These results have long been established for self-adjoint operators of the form  $(Lu)(x) = e^{V(x)} \nabla \cdot [e^{-V(x)} \mathbb{S}(x) \nabla u(x)]$ , provided  $\mathbb{S}(x)$  are smooth, positive-definite, symmetric matrices, and  $V$  is a smooth potential. They have been extended, more

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recently, by Pinsky [19, 21] to the case in which the operator  $L$  is not self-adjoint. In this situation, the minimization formula for the capacity, mentioned above, has to be replaced by a minmax problem.

The first main result of this article provides two variational formulae for the capacity (1.2) in terms of divergence-free flows. In contrast with the minmax formulae, the first optimization problem is expressed as an infimum, while the second one is expressed as a supremum, simplifying the task of obtaining lower and upper bounds for the capacity.

Analogous Dirichlet and Thomson principle were obtained by Gaudillière and Landim [13] (the Dirichlet principle) and by Slowik [22] (the Thomson principle) for continuous-time Markov chains.

In the second part of the article, we use the formulae for the capacity to examine the metastable behavior of a non-reversible diffusion in a double well potential.

Let  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  be a smooth, double-well potential which diverges at infinity, and let  $\mathbb{M}$  be a non-symmetric, positive-definite matrix. We impose in the next section further assumptions on  $U$ . Denote by  $\mathbb{M}^\dagger$  the transpose of  $\mathbb{M}$  and by  $\mathbb{S} = (\mathbb{M} + \mathbb{M}^\dagger)/2$  its symmetric part. Consider the diffusion  $X_t^\epsilon$ ,  $\epsilon > 0$ , described by the SDE

$$dX_t^\epsilon = -\mathbb{M}(\nabla U)(X_t^\epsilon) dt + \sqrt{2\epsilon} \mathbb{K} dW_t, \quad (1.3)$$

where  $W_t$  is a standard  $d$ -dimensional Brownian motion, and  $\mathbb{K}$  is the symmetric, positive-definite square root of  $\mathbb{S}$ ,  $\mathbb{S} = \mathbb{K}\mathbb{K}$ .

Assume that  $U$  has two local minima, denoted by  $\mathbf{m}_1, \mathbf{m}_2$ , separated by a single saddle point  $\boldsymbol{\sigma}$ , and that  $U(\mathbf{m}_2) \leq U(\mathbf{m}_1)$ . The stationary state of  $X_t^\epsilon$ , given by  $\mu_\epsilon(dx) \sim \exp\{-U(x)/\epsilon\} dx$ , is concentrated in a neighborhood of  $\mathbf{m}_2$  when the previous inequality is strict.

If  $X_t^\epsilon$  starts from a neighborhood of  $\mathbf{m}_1$ , it remains there for a long time in the small noise limit  $\epsilon \rightarrow 0$  until it overcomes the potential barrier and jumps to a neighborhood of  $\mathbf{m}_2$  through the saddle point  $\boldsymbol{\sigma}$ . Denote by  $\tau_\epsilon$  the hitting time of a neighborhood of  $\mathbf{m}_2$ . The asymptotic behavior of the mean value of  $\tau_\epsilon$  as  $\epsilon \rightarrow 0$  has been the object of many studies.

The Arrhenius' law [3] asserts that the mean value is logarithmic equivalent to the potential barrier:  $\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}_{\mathbf{m}_1}[\tau_\epsilon] = U(\boldsymbol{\sigma}) - U(\mathbf{m}_1) =: \Delta U$ , where  $\mathbb{E}_{\mathbf{m}_1}$  represents the expectation of the diffusion  $X_t^\epsilon$  starting from  $\mathbf{m}_1$ . The sub-exponential corrections, known as the Eyring-Kramers formula [10, 16], have been computed when the matrix  $\mathbb{M}$  is symmetric and the potential non-degenerate at the critical points. Assuming that the Hessian of the potential is positive definite at  $\mathbf{m}_1$  and that it has a unique negative eigenvalue at  $\boldsymbol{\sigma}$ , denoted by  $-\lambda$ , while all the others are strictly positive, the sub-exponential prefactor is given by

$$\mathbb{E}_{\mathbf{m}_1}[\tau_\epsilon] = [1 + o_\epsilon(1)] \frac{2\pi}{\lambda} \frac{\sqrt{-\det[(\text{Hess } U)(\boldsymbol{\sigma})]}}{\sqrt{\det[(\text{Hess } U)(\mathbf{m}_1)]}} e^{\Delta U/\epsilon},$$

where  $o_\epsilon(1) \rightarrow 0$  as  $\epsilon$  vanishes.

This estimate appears in articles published in the 60's. A rigorous proof was first obtained by Bovier, Eckhoff, Gaynard, and Klein [8] with arguments from potential theory, and right after by Helffer, Klein and Nier [14] through Witten Laplacian analysis. We refer to Berglund [6] and Bouchet and Reygner [7] for an historical overview and further references.

Recently, Bouchet and Reygner [7] extended the Eyring-Kramers formula to the non-reversible setting. They showed that in this context the negative eigenvalue  $-\lambda$  has to be replaced by the unique negative eigenvalue of  $(\text{Hess } U)(\sigma) \mathbb{M}$ .

We present below a rigorous proof of this result, based on the variational formulae obtained for the capacity in the first part of the article, and on the approach developed by Bovier, Eckhoff, Gaynard, and Klein [8] in the reversible case. This estimate permits to describe the metastable behavior of the diffusion  $X_t^\varepsilon$  in the small noise limit. Analogous results have been derived for random walks in a potential field in [17, 18].

## 2. NOTATION AND RESULTS

We start by introducing the main assumptions. We frequently refer to [12] for results on elliptic equations and to [11, 21] for results on diffusions.

**2.1. A Dirichlet and a Thomson principle.** Fix  $d \geq 2$ , and denote by  $C^k(\mathbb{R}^d)$ ,  $0 \leq k \leq \infty$ , the space of real functions on  $\mathbb{R}^d$  whose partial derivatives up to order  $k$  are continuous. Let  $\mathbb{M}_{m,n}$ ,  $1 \leq m, n \leq d$ , be functions in  $C^2(\mathbb{R}^d)$  for which there exists a finite constant  $C_0$  such that

$$\sum_{m,n=1}^d \mathbb{M}_{m,n}(x)^2 \leq C_0 \quad \text{for all } x \in \mathbb{R}^d. \quad (2.1)$$

Denote by  $\mathbb{M}(x)$  the matrix whose entries are  $\mathbb{M}_{m,n}(x)$ . Assume that the matrices  $\mathbb{M}(x)$ ,  $x \in \mathbb{R}^d$ , are uniformly positive-definite: There exist  $0 < \lambda < \Lambda$  such that for all  $x, \xi \in \mathbb{R}^d$ ,

$$\lambda \|\xi\|^2 \leq \xi \cdot \mathbb{M}(x) \xi \leq \Lambda \|\xi\|^2, \quad (2.2)$$

where  $\eta \cdot \xi$  represents the scalar product in  $\mathbb{R}^d$ , and  $\|\xi\|$  the Euclidean norm.

Let  $V$  be a function in  $C^3(\mathbb{R}^d)$  such that  $\int_{\mathbb{R}^d} \exp\{-V(x)\} dx < \infty$ , and assume, without loss of generality, that  $\int_{\mathbb{R}^d} \exp\{-V(x)\} dx = 1$ . Denote by  $\mu$  the probability measure on  $\mathbb{R}^d$  defined by  $\mu(dx) = \exp\{-V(x)\} dx$ .

Denote by  $\mathcal{L}$  the differential operator which acts on functions in  $C^2(\mathbb{R}^d)$  as

$$(\mathcal{L}f)(x) = e^{V(x)} \nabla \cdot \{e^{-V(x)} \mathbb{M}(x) (\nabla f)(x)\}.$$

In this formula,  $\nabla g$  represents the gradient of a function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\nabla \cdot \Phi$  the divergence of a vector field  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . The previous formula can be rewritten as

$$(\mathcal{L}f)(x) = \sum_{j=1}^d b_j(x) \partial_{x_j} f(x) + \sum_{j,k=1}^d \mathbb{S}_{j,k}(x) \partial_{x_j, x_k}^2 f(x), \quad (2.3)$$

where the drift  $b = (b_1, \dots, b_d)$  is given by

$$b_j(x) = \sum_{k=1}^d \left\{ \partial_{x_k} \mathbb{M}_{k,j}(x) - (\partial_{x_k} V)(x) \mathbb{M}_{k,j}(x) \right\},$$

and where  $\mathbb{S}(x)$  represents the symmetric part of the matrix  $\mathbb{M}(x)$ ,  $\mathbb{S}(x) = (1/2)[\mathbb{M}(x) + \mathbb{M}^\dagger(x)]$ ,  $\mathbb{M}^\dagger(x)$  being the transpose of  $\mathbb{M}(x)$ .

Denote by  $B(r)$ ,  $r > 0$ , the open ball of radius  $r$  centered at the origin, and by  $\partial B(r)$  its boundary. We assume that

$$\lim_{n \rightarrow \infty} \inf_{z \notin B(n)} V(z) = \infty, \quad (2.4)$$

and that there exist  $r_1 > 0$ ,  $c_1 > 0$  such that

$$(\mathcal{L}V)(x) \leq -c_1 \quad (2.5)$$

for all  $x$  such that  $\|x\| \geq r_1$ . By (2.3), this last condition can be rewritten as

$$\begin{aligned} \sum_{j,k=1}^d (\partial_{x_j} \mathbb{M}_{j,k})(x) (\partial_{x_k} V)(x) + \sum_{j,k=1}^d \mathbb{S}_{j,k}(x) \partial_{x_j, x_k}^2 V(x) + c_1 \\ \leq (\nabla V)(x) \cdot \mathbb{S}(x) (\nabla V)(x) \end{aligned}$$

for all  $x$  such that  $\|x\| \geq r_1$ .

It follows from the first condition in (2.4) that  $V$  is bounded below by a finite constant: there exists  $c_2 \in \mathbb{R}$  such that  $V(y) \geq c_2$  for all  $y \in \mathbb{R}^d$ . Of course,  $\mathbb{M}(x) = \mathbb{I}$ , where  $\mathbb{I}$  represents the identity matrix, and  $V(x) = \|x\|^2 + c$  satisfy all previous hypotheses for an appropriate constant  $c$ .

The regularity of  $\mathbb{M}$  and  $V$ , and assumptions (2.1), (2.2) are sufficient to guarantee the existence of smooth solutions of some Dirichlet problems. Conditions (2.4), (2.5) guarantee that the process whose generator is given by  $\mathcal{L}$  is positive recurrent.

**Elliptic equations** Fix a domain (open and connected set)  $\Omega \subseteq \mathbb{R}^d$ . Denote by  $C^k(\Omega)$ ,  $k \geq 0$ , the space of functions on  $\Omega$  whose partial derivatives up to order  $k$  are continuous, and by  $C^{k,\alpha}(\Omega)$ ,  $0 < \alpha < 1$ , the space of function in  $C^k(\Omega)$  whose  $k$ -th order partial derivatives are Hölder continuous with exponent  $\alpha$ .

Denote by  $\overline{\Omega}$  the closure of  $\Omega$  and by  $\partial\Omega$  its boundary. The domain  $\Omega$  is said to have a  $C^{k,\alpha}$ -boundary, if for each point  $x \in \partial\Omega$ , there is a ball  $B$  centered at  $x$  and a one-to-one map  $\psi$  from  $B$  onto  $C \subset \mathbb{R}^d$  such that  $\psi(B \cap \Omega) \subset \{z \in \mathbb{R}^d : z_d > 0\}$ ,  $\psi(B \cap \partial\Omega) \subset \{z \in \mathbb{R}^d : z_d = 0\}$ ,  $\psi \in C^{k,\alpha}(B)$ ,  $\psi^{-1} \in C^{k,\alpha}(C)$ .

Denote by  $L^2(\Omega)$  the space of functions  $f : \Omega \rightarrow \mathbb{R}$  endowed with the scalar product  $\langle \cdot, \cdot \rangle_\mu$  given by

$$\langle f, g \rangle_\mu = \int_\Omega f g d\mu,$$

and by  $W^{1,2}(\Omega)$  the Hilbert space of weakly differentiable functions endowed with the scalar product  $\langle f, g \rangle_1$  given by

$$\langle f, g \rangle_1 = \int_\Omega f g d\mu + \int_\Omega \nabla f \cdot \nabla g d\mu.$$

Fix  $0 < \alpha < 1$ , a function  $\mathbf{g}$  in  $L^2(\Omega) \cap C^\alpha(\overline{\Omega})$  and a function  $\mathbf{b}$  in  $W^{1,2}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ . Assume that  $\Omega$  has a  $C^{2,\alpha}$ -boundary. It follows from assumptions (2.1), (2.2) and Theorems 8.3, 8.8 and 9.19 in [12] that the Dirichlet boundary-value problem

$$\begin{cases} (\mathcal{L}u)(x) = -\mathbf{g}(x) & x \in \Omega, \\ u(x) = \mathbf{b}(x) & x \in \partial\Omega. \end{cases} \quad (2.6)$$

has a unique solution in  $W^{1,2}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ . Moreover, by the maximum principle, [12, Theorem 8.1], if  $\mathbf{g} = 0$ ,

$$\inf_{x \in \partial\Omega} \mathbf{b}(x) \leq \inf_{y \in \Omega} u(y) \leq \sup_{y \in \Omega} u(y) \leq \sup_{x \in \partial\Omega} \mathbf{b}(x). \quad (2.7)$$

The proofs of Theorems 8.3 and 8.8 in [12] require simple modifications since it is easier to work with  $\mu(dx)$  as reference measure than the Lebesgue measure.

**Dirichlet and Thomson principle** We will frequently assume that a pair of sets  $A, B$  with  $C^1$ -boundaries fulfill the following conditions. Denote by  $d(A, B)$  the

distance between the sets  $A, B$ ,  $d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$ , and by  $\sigma(\partial A)$  the measure of the boundary of  $A$ .

**Assumption S.** The sets  $A, B$  are two bounded domains of  $\mathbb{R}^d$  with  $C^{2,\alpha}$ -boundaries, for some  $0 < \alpha < 1$ , and finite perimeter,  $\sigma(A) < \infty$ ,  $\sigma(B) < \infty$ . Moreover,  $d(A, B) > 0$ , and the set  $\Omega = (\bar{A} \cup \bar{B})^c$  is a domain.

Denote by  $h_{A,B}$  the unique solution of the Dirichlet problem (2.6) with  $\Omega = (\bar{A} \cup \bar{B})^c$ ,  $\mathbf{g} = 0$ , and  $\mathbf{b}$  such that  $\mathbf{b}(x) = 1, 0$  if  $x \in \partial A, \partial B$ , respectively. The function  $h_{A,B}$  is called the *equilibrium potential* between  $A$  and  $B$ . Similarly, denote by  $h_{A,B}^*$  the solution to (2.6) with  $\mathbb{M}$  replaced by its transpose  $\mathbb{M}^\dagger$  and the same functions  $\mathbf{g}$  and  $\mathbf{b}$ .

The capacity between  $A$  and  $B$ , denoted by  $\text{cap}(A, B)$ , is defined as

$$\text{cap}(A, B) = \int_{\partial A} [\mathbb{M}(x) \nabla h_{A,B}(x)] \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx), \quad (2.8)$$

where  $\sigma(dx)$  represents the surface measure on the boundary  $\partial\Omega$  and  $\mathbf{n}_\Omega$  represents the outward normal vector to  $\partial\Omega$  (and, therefore, the inward normal vector to  $\partial A \cup \partial B$ ). Since  $\partial A$  is the 1-level set of the equilibrium potential  $h_{A,B}$  which, by the maximum principle, is bounded by 1,  $\nabla h_{A,B} = c \mathbf{n}_\Omega(x)$  for some  $c \geq 0$  so that  $\mathbb{M}(x) (\nabla h_{A,B})(x) \cdot \mathbf{n}_\Omega(x) = \mathbb{S}(x) (\nabla h_{A,B})(x) \cdot \mathbf{n}_\Omega(x) \geq 0$ . The capacity can also be expressed as

$$\text{cap}(A, B) = - \int_{\partial A} \frac{\partial h_{A,B}}{\partial v} e^{-V} d\sigma,$$

where  $v(x) = \mathbb{M}^\dagger(x) \mathbf{n}_A(x)$ . We present in Section 3 some properties of the capacity.

Let  $\mathcal{F} = \mathcal{F}_{A,B}$  be the Hilbert space of vector fields  $\varphi : \Omega \rightarrow \mathbb{R}^d$  endowed with the scalar product given by

$$\langle \varphi, \psi \rangle = \int_{\Omega} \varphi(x) \cdot \mathbb{S}(x)^{-1} \psi(x) e^{V(x)} dx.$$

By assumption (2.2),  $\langle \varphi, \varphi \rangle \leq \lambda^{-1} \int_{\Omega} \|\varphi(x)\|^2 e^{V(x)} dx$ . By Schwarz inequality, for every  $\varphi, \psi \in \mathcal{F}$ ,

$$\langle \varphi, \psi \rangle^2 \leq \langle \varphi, \varphi \rangle \langle \psi, \psi \rangle. \quad (2.9)$$

Denote by  $\mathcal{F}^{(c)}$ ,  $c \in \mathbb{R}$ , the space of vector fields  $\varphi \in \mathcal{F}$  of class  $C^1(\bar{\Omega})$  such that

$$\begin{aligned} (\nabla \cdot \varphi)(x) &= 0 \quad \text{for } x \in \Omega, \\ - \int_{\partial A} \varphi(x) \cdot \mathbf{n}_\Omega(x) \sigma(dx) &= c = \int_{\partial B} \varphi(x) \cdot \mathbf{n}_\Omega(x) \sigma(dx), \end{aligned}$$

The reason for the minus sign is due to the convention that  $\mathbf{n}_\Omega(x)$  is the inward normal to  $\partial A$ . The integrals over  $\partial A, \partial B$  are well defined because  $\varphi$  is continuous, and  $A, B$  have finite perimeter. On the other hand, the integral over  $\partial B$  must be equal to minus the one over  $\partial A$  because  $\varphi$  is divergence free on  $\Omega$ .

For a function  $f : \bar{\Omega} \rightarrow \mathbb{R}$  in  $C^2(\bar{\Omega}) \cap W^{1,2}(\Omega)$ , denote by  $\Phi_f, \Phi_f^*, \Psi_f$  the elements of  $\mathcal{F}$  given by

$$\Psi_f = e^{-V} \mathbb{S} \nabla f, \quad \Phi_f = e^{-V} \mathbb{M}^\dagger \nabla f, \quad \Phi_f^* = e^{-V} \mathbb{M} \nabla f. \quad (2.10)$$

Denote by  $\mathcal{C}_{A,B}^{a,b}$ ,  $a, b \in \mathbb{R}$  the set of bounded functions  $f$  in  $C^2(\bar{\Omega}) \cap W^{1,2}(\Omega)$  such that  $f(x) = a$ ,  $x \in \partial A$ ,  $f(y) = b$ ,  $y \in \partial B$ .

**Proposition 2.1** (Dirichlet Principle). *Let  $A, B$  be two open subsets satisfying Assumption S. Then,*

$$\text{cap}(A, B) = \inf_{f \in \mathcal{C}_{A,B}^{1,0}} \inf_{\varphi \in \mathcal{F}^{(1)}} \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle. \quad (2.11)$$

*The minimum is attained at  $f = (1/2)(h_{A,B} + h_{A,B}^*)$ , and  $\varphi = \Phi_f - \Psi_{h_{A,B}}$ .*

**Proposition 2.2** (Thomson principle). *Let  $A, B$  be two open subsets satisfying Assumptions S. Then,*

$$\text{cap}(A, B) = \sup_{f \in \mathcal{C}_{A,B}^{0,0}} \sup_{\varphi \in \mathcal{F}^{(1)}} \frac{1}{\langle \Phi_f - \varphi, \Phi_f - \varphi \rangle}. \quad (2.12)$$

*The maximum is attained at  $f = (h_{A,B} - h_{A,B}^*)/2 \text{cap}(A, B)$ , and  $\varphi = \Phi_f - \Psi_{g_{A,B}}$ , where  $g_{A,B} = h_{A,B}/\text{cap}(A, B)$ .*

In section 4, we present a Dirichlet and a Thomson principle in compact manifolds.

**2.2. Diffusions in double-well potential field.** In this subsection, we state the Eyring-Kramers formula for a non-reversible diffusion in a double-well potential field.

Consider a potential  $U : \mathbb{R}^d \rightarrow \mathbb{R}$ . Denote by  $H_{\mathbf{x}, \mathbf{y}}$  the height of the saddle points between  $\mathbf{x}$  and  $\mathbf{y} \in \mathbb{R}^d$ :

$$H_{\mathbf{x}, \mathbf{y}} = \inf_{\gamma} H(\gamma) := \inf_{\gamma} \sup_{\mathbf{z} \in \gamma} U(\mathbf{z}), \quad (2.13)$$

where the infimum is carried over the set  $\Gamma_{\mathbf{x}, \mathbf{y}}$  of all continuous paths  $\gamma : [0, 1] \rightarrow \mathbb{R}^d$  such that  $\gamma(0) = \mathbf{x}$ ,  $\gamma(1) = \mathbf{y}$ . Let  $G_{\mathbf{x}, \mathbf{y}}$  be the smallest subset of  $\{\mathbf{z} \in \mathbb{R}^d : U(\mathbf{z}) = H_{\mathbf{x}, \mathbf{y}}\}$  with the property that any path  $\gamma \in \Gamma_{\mathbf{x}, \mathbf{y}}$  such that  $H(\gamma) = H_{\mathbf{x}, \mathbf{y}}$  contains a point in  $G_{\mathbf{x}, \mathbf{y}}$ . The set  $G_{\mathbf{x}, \mathbf{y}}$  is called the set of gates between  $\mathbf{x}$  and  $\mathbf{y}$ .

**The Potential.** We assume that the potential field  $U$  is such that

- (P1)  $U \in C^3(\mathbb{R}^d)$  and  $\lim_{n \rightarrow \infty} \inf_{\mathbf{x} : \|\mathbf{x}\| \geq n} U(\mathbf{x}) = \infty$ .
- (P2) The function  $U$  has finitely many critical points. Only two of them, denoted by  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , are local minima. The Hessian of  $U$  at each of these minima has  $d$  strictly positive eigenvalues.
- (P3) The set of gates between  $\mathbf{m}_1$  and  $\mathbf{m}_2$  is formed by  $\ell \geq 1$  saddle points, denoted by  $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_\ell$ . The Hessian of  $U$  at each saddle point  $\boldsymbol{\sigma}_i$  has exactly one strictly negative eigenvalue and  $(d - 1)$  strictly positive eigenvalues.
- (P4) The function  $U$  satisfies

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \nabla U(\mathbf{x}) = \lim_{\|\mathbf{x}\| \rightarrow \infty} \left\{ \|\nabla U(\mathbf{x})\| - 2\Delta U(\mathbf{x}) \right\} = \infty, \quad (2.14)$$

and

$$Z_\epsilon := \int_{\mathbb{R}^d} \exp\{-U(\mathbf{x})/\epsilon\} d\mathbf{x} < \infty$$

for all  $\epsilon > 0$ .

It is not difficult to show that the conditions (2.14) imply that

$$\int_{\mathbf{x} : U(\mathbf{x}) \geq a} e^{-U(\mathbf{x})/\epsilon} d\mathbf{x} \leq C(a) e^{-a/\epsilon} \quad (2.15)$$

where the constant  $C(a)$  is uniform in  $\epsilon \leq 1$ .

**Diffusion model.** Let  $\mathbb{M}$  be a  $d \times d$  (generally non-symmetric) positive-definite matrix:  $\mathbf{v} \cdot \mathbb{M} \mathbf{v} > 0$  for all  $\mathbf{v} \neq \mathbf{0}$ . Denote by  $\{X_t^\epsilon : t \in [0, \infty)\}$ ,  $\epsilon > 0$ , the diffusion process associated to the generator  $\mathcal{L}_\epsilon$  given by

$$(\mathcal{L}_\epsilon f)(\mathbf{x}) = -\nabla U(\mathbf{x}) \cdot \mathbb{M}(\nabla f)(\mathbf{x}) + \epsilon \sum_{1 \leq i, j \leq d} \mathbb{M}_{ij}(\partial_{x_i, x_j}^2 f)(\mathbf{x}).$$

Note that we can rewrite the generator  $\mathcal{L}_\epsilon$  as

$$(\mathcal{L}_\epsilon f)(\mathbf{x}) = \epsilon e^{U(\mathbf{x})/\epsilon} \nabla \cdot \left[ e^{-U(\mathbf{x})/\epsilon} \mathbb{M}(\nabla f)(\mathbf{x}) \right].$$

In particular, the probability measure

$$\mu_\epsilon(d\mathbf{x}) := Z_\epsilon^{-1} \exp\{-U(\mathbf{x})/\epsilon\} d\mathbf{x}$$

is the stationary state of the process  $X_t^\epsilon$ .

The process  $X_t^\epsilon$  can also be written as the solution of a stochastic differential equation. Recall that  $\mathbb{K}$  represents the symmetric, positive-definite square root of the symmetric matrix  $\mathbb{S} = (\mathbb{M} + \mathbb{M}^\dagger)/2$ :  $\mathbb{S} = \mathbb{K}\mathbb{K}$ . It is easy to check that  $X_t^\epsilon$  is the solution of the stochastic differential equation (1.3).

Let  $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^d$  be two open subsets of  $\mathbb{R}^d$  satisfying the assumptions S, and let  $\Omega = (\overline{\mathcal{A}} \cup \overline{\mathcal{B}})^c$ . In the present context, the capacity, defined in (2.8), is given by

$$\text{cap}(\mathcal{A}, \mathcal{B}) = \frac{\epsilon}{Z_\epsilon} \int_{\partial \mathcal{A}} [\mathbb{M}(\mathbf{x}) \nabla h_{\mathcal{A}, \mathcal{B}}(\mathbf{x})] \cdot \mathbf{n}_\Omega(\mathbf{x}) e^{-U(\mathbf{x})/\epsilon} \sigma(d\mathbf{x}). \quad (2.16)$$

**Structure of valleys.** Let  $h_i = U(\mathbf{m}_i)$ ,  $i = 1, 2$ , and assume without loss of generality that  $h_1 \geq h_2$ , so that  $\mathbf{m}_2$  is the global minimum of the potential  $U$ . Denote by  $H$  the height of the saddle points  $\mathfrak{S} := \{\sigma_1, \dots, \sigma_\ell\}$ :

$$H := U(\sigma_1) = \dots = U(\sigma_\ell).$$

Let  $\Omega$  be the level set defined by saddle points which separate  $\mathbf{m}_1$  from  $\mathbf{m}_2$ :

$$\Omega := \{\mathbf{x} \in \mathbb{R}^d : U(\mathbf{x}) < H\}.$$

Denote by  $\mathcal{W}_1$  and  $\mathcal{W}_2$  the two connected components of  $\Omega$  such that  $\mathbf{m}_i \in \mathcal{W}_i$ ,  $i = 1, 2$ , respectively. Note that  $\overline{\mathcal{W}_1} \cap \overline{\mathcal{W}_2} = \mathfrak{S}$ .

Denote by  $\mathcal{V}_1$  and  $\mathcal{V}_2$  two metastable sets containing  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , respectively. More precisely,  $\mathcal{V}_i$ ,  $i = 1, 2$ , is a open subset of  $\mathcal{W}_i$  which satisfies assumptions S and such that

$$B_\epsilon(\mathbf{m}_i) \subset \mathcal{V}_i \subset \{\mathbf{x} \in \mathbb{R}^d : U(\mathbf{x}) < U(\sigma) - \kappa\}$$

for some  $\kappa > 0$ , where  $B_\epsilon(\mathbf{m}_i)$  represents the ball of radius  $\epsilon$  centered at  $\mathbf{m}_i$ :  $B_\epsilon(\mathbf{m}_i) = \{\mathbf{x} : |\mathbf{x} - \mathbf{m}_i| < \epsilon\}$ .

**Metastability results.** Fix a saddle point  $\sigma$  of the potential  $U$ . Denote by  $-\lambda_1^\sigma < 0 < \lambda_2^\sigma < \dots < \lambda_d^\sigma$  the eigenvalues of  $(\text{Hess } U)(\sigma) := \mathbb{L}^\sigma$ . By Lemma 10.1 of [17], both  $\mathbb{L}^\sigma \mathbb{M}$  and  $\mathbb{L}^\sigma \mathbb{M}^\dagger$  have a unique negative eigenvalue. The negative eigenvalues of  $\mathbb{L}^\sigma \mathbb{M}$  and  $\mathbb{L}^\sigma \mathbb{M}^\dagger$  coincide because  $\mathbb{L}^\sigma \mathbb{M}^\dagger = \mathbb{L}^\sigma (\mathbb{L}^\sigma \mathbb{M})^\dagger (\mathbb{L}^\sigma)^{-1}$ . Denote by  $-\mu^\sigma$  this common negative eigenvalue, and let

$$\omega(\sigma) := \frac{\mu^\sigma}{\sqrt{-\det[(\text{Hess } U)(\sigma)]}}; \quad \sigma \in \mathfrak{S}. \quad (2.17)$$

We prove in Section 5 the following sharp estimate for capacity between the valleys  $\mathcal{V}_1$  and  $\mathcal{V}_2$ .

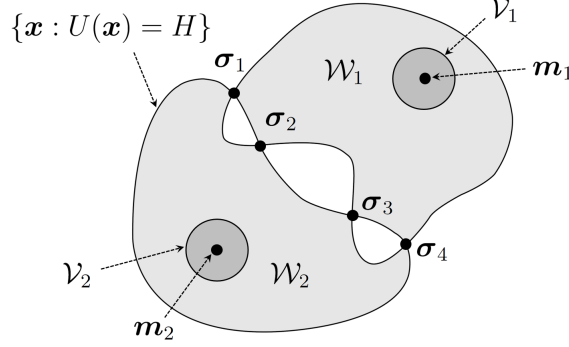


FIGURE 1. The structure of metastable wells and valleys.

**Theorem 2.3.** *We have the following estimate on the capacity.*

$$\text{cap}(\mathcal{V}_1, \mathcal{V}_2) = [1 + o_\epsilon(1)] \frac{1}{Z_\epsilon} e^{-H/\epsilon} \frac{(2\pi\epsilon)^{d/2}}{2\pi} \sum_{i=1}^{\ell} \omega(\sigma_i). \quad (2.18)$$

The metastable behavior of  $X_t^\epsilon$  follows from this result. In Section 6, we derive a sharp estimate for the transitions time between the two different wells stated below.

Denote by  $\mathbb{P}_x$ ,  $x \in \mathbb{R}^d$ , the probability measure on  $C(\mathbb{R}_+, \mathbb{R}^d)$  induced by the Markov process  $X_t^\epsilon$  starting from  $x$ . Expectation with respect to  $X_t$  is represented by  $\mathbb{E}_x$ .

Denote by  $H_C$ ,  $C$  an open subset of  $\mathbb{R}^d$ , the hitting time of the set  $C$ :

$$H_C = \inf\{t \geq 0 : X_t \in C\}. \quad (2.19)$$

**Theorem 2.4.** *Under the notations above,*

$$\mathbb{E}_{\mathbf{m}_1}[H_{\mathcal{V}_2}] = [1 + o_\epsilon(1)] \frac{2\pi e^{(H-h_1)/\epsilon}}{\sqrt{\det[(\text{Hess } U)(\mathbf{m}_1)]}} \left( \sum_{i=1}^{\ell} \omega(\sigma_i) \right)^{-1}. \quad (2.20)$$

**Remark 2.5.** *Let  $\Xi \subset \mathbb{R}^d$  be a bounded domain with a boundary in  $C^{2,\alpha}$  for some  $0 < \alpha < 1$ . Assume that the potential has no critical points at  $\partial\Xi$  and that  $\mathbf{n}_\Xi \cdot \nabla U > 0$  at  $\partial\Xi$ . A similar result can be proven for a diffusion evolving on  $\Xi$  with Neumann boundary conditions.*

**Remark 2.6.** *Theorems 2.3, 2.4 together with the theory developed in [4, 5] permit to describe the metastable behavior of the diffusion  $X_t^\epsilon$ . Denote by  $\theta_\epsilon$  the expression multiplied by  $[1 + o_\epsilon(1)]$  in (2.20). If  $U(\mathbf{m}_2) < U(\mathbf{m}_1)$ , on the time scale  $\theta_\epsilon$ , starting from  $\mathbf{m}_1$ , the diffusion process  $X_t^\epsilon$  remains a mean 1 exponential time in a neighborhood of  $\mathbf{m}_1$ , after which it jumps to a neighborhood of  $\mathbf{m}_2$  and there remains for ever. If  $U(\mathbf{m}_2) = U(\mathbf{m}_1)$ , on the time scale  $\theta_\epsilon$ ,  $X_t^\epsilon$  behaves as a two-state Markov chain which jumps from one state to the other at mean 1 exponential times.*

**Remark 2.7.** *The arguments presented in the next sections to prove Theorems 2.3 and 2.4 apply to the case in which the entries of the matrix  $\mathbb{M}(\mathbf{x})$  belong to  $C^2(\mathbb{R}^d)$  and satisfy conditions (2.1), (2.2).*



The article is organized as follows. Sections 3 and 4 are devoted to Propositions 2.1 and 2.2 and to the extension of the Dirichlet and the Thomson principle for elliptic operators in compact manifolds without boundaries. In section 5, we prove Theorem 2.3 by constructing vector fields which approximate the optimal ones. The properties of these vector fields are derived in Section 7, based on general estimates presented in Section 6. The last section is dedicated to Theorem 2.4.

### 3. THE DIRICHLET AND THE THOMSON PRINCIPLES

Denote by  $C([0, \infty), \mathbb{R}^d)$  the space of continuous functions  $\omega$  from  $\mathbb{R}_+$  to  $\mathbb{R}^d$  endowed with the topology of uniform convergence on bounded intervals. Let  $X_t$ ,  $t \geq 0$  be the one-dimensional projections:  $X_t(\omega) = \omega(t)$ ,  $\omega \in C([0, \infty), \mathbb{R}^d)$ . We sometimes represent  $X_t$  as  $X(t)$ .

It follows from the assumptions on  $\mathbb{M}$  and  $V$ , and from Theorems 1.10.4 and 1.10.6 in [21], that there exists a unique solution, denoted hereafter by  $\{\mathbb{P}_x : x \in \mathbb{R}^d\}$ , to the martingale problem associated to the generator  $\mathcal{L}$ . Moreover, the family  $\{\mathbb{P}_x : x \in \mathbb{R}^d\}$  possesses the strong Markov and the Feller properties. Expectation with respect to  $\mathbb{P}_x$  is expressed as  $\mathbb{E}_x$ .

The process  $X_t$  can be represented in terms of a stochastic differential equation. Denote by  $\Sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$  the square root of  $\mathbb{S}(x)$ , in the sense that  $\Sigma(x)$  is a positive-definite, symmetric matrix such that  $\Sigma(x)\Sigma(x) = \mathbb{S}(x)$ . By [11, Lemma 6.1.1], the entries of  $\Sigma$  inherit the regularity properties of  $\mathbb{S}$ :  $\Sigma_{m,n}$  belongs to  $C^3(\mathbb{R}^d)$  for  $1 \leq m, n \leq d$ . The process  $X_t$  is the unique solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sqrt{2}\Sigma(X_t)dB_t,$$

where  $B_t$  stands for a  $d$ -dimensional Brownian motion.

In view of condition (2.5), by [21, Theorem 6.1.3], the process  $X_t$  is positive recurrent. Furthermore, by [21, Theorem 4.9.6],  $\mathbb{E}_x[H_C] < \infty$  for all open sets  $C$  and all  $x \notin C$ . Finally, an elementary computation shows that the probability measure  $\mu$  is stationary.

The solutions of the elliptic equation (2.6) can be represented in terms of the process  $X_t$ . Fix  $0 < \alpha < 1$ , a bounded function  $\mathbf{g}$  in  $L^2(\Omega) \cap C^\alpha(\overline{\Omega})$  and a bounded function  $\mathbf{b}$  in  $W^{1,2}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ . Assume that  $\Omega$  has a  $C^{2,\alpha}$ -boundary. It follows from the proof of [11, Theorem 6.5.1] and from the positive recurrence that the unique solution  $u$  of (2.6) can be represented as

$$u(x) = \mathbb{E}_x[\mathbf{b}(X(H_{\Omega^c}))] + \mathbb{E}_x\left[\int_0^{H_{\Omega^c}} \mathbf{g}(X_t)dt\right]. \quad (3.1)$$

In particular if  $A, B$  represent two open sets satisfying the assumptions S, the equilibrium potential between  $A$  and  $B$ , introduced just above (2.8), is given by

$$h_{A,B}(x) = \mathbb{P}_x[H_A < H_B]. \quad (3.2)$$

**3.1. Properties of the capacity.** We present in this subsection some elementary properties of the capacity. We begin with an alternative formula for the capacity. Unless otherwise stated, until the end of this section, the open subsets  $A, B$  satisfy the assumptions S.

**Lemma 3.1.** *We have that*

$$\text{cap}(A, B) = \int_{\mathbb{R}^d} \nabla h_{A,B}(x) \cdot \mathbb{S}(x) \nabla h_{A,B}(x) \mu(dx).$$

*Proof.* Since the function  $h_{A,B}$  is harmonic on  $\Omega = (\overline{A} \cup \overline{B})^c$ , and since it is equal to 1 on the set  $\partial A$  and 0 on the set  $\partial B$ , the capacity  $\text{cap}(A, B)$  can be written as

$$\begin{aligned} & \int_{\partial A} h_{A,B}(x) [\mathbb{M}(x) \nabla h_{A,B}(x)] \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx) \\ & + \int_{\partial B} h_{A,B}(x) [\mathbb{M}(x) \nabla h_{A,B}(x)] \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx) \\ & - \int_{\Omega} h_{A,B}(x) \nabla \cdot [e^{-V(x)} \mathbb{M}(x) \nabla h_{A,B}(x)] dx . \end{aligned}$$

Since  $h_{A,B}$  belongs to  $C^{2+\alpha}(\overline{\Omega}) \cap W^{1,2}(\Omega)$  and since  $\mathbb{S}$  represents the symmetric part of the matrix  $\mathbb{M}$ , by the divergence theorem, the previous expression is equal to

$$\int_{\Omega} \nabla h_{A,B}(x) \mathbb{S}(x) \nabla h_{A,B}(x) \mu(dx) .$$

As the equilibrium potential is constant in  $A \cup B$ , we may replace in the previous integral the domain  $\Omega$  by  $\mathbb{R}^d$ , which completes the proof of the lemma.  $\square$

Since  $h_{B,A} = 1 - h_{A,B}$ , it follows from the previous lemma that the capacity is symmetric: for every disjoint subsets  $A, B$  of  $\mathbb{R}^d$ ,

$$\text{cap}(A, B) = \text{cap}(B, A) . \quad (3.3)$$

**Adjoint generator.** Denote by  $\mathcal{L}^*$  the  $L^2(\mu)$  adjoint of the generator  $\mathcal{L}$ , which acts on functions in  $C^2(\mathbb{R}^d)$  as

$$(\mathcal{L}^* f)(x) = e^{V(x)} \nabla \cdot \{e^{-V(x)} \mathbb{M}^\dagger(x) (\nabla f)(x)\} .$$

Let  $\mathcal{S}$  be the symmetric part of the generator  $\mathcal{L}$ , defined as  $\mathcal{S} = (1/2)(\mathcal{L} + \mathcal{L}^*)$  and acting on  $C^2(\mathbb{R}^d)$  as  $\mathcal{S}f = e^V \nabla \cdot (e^{-V} \mathbb{S} \nabla f)$ .

Denote by  $\text{cap}^*(A, B)$  the capacity between the open sets  $A, B$  with respect to the adjoint generator  $\mathcal{L}^*$ . In view of (2.8), this capacity  $\text{cap}^*(A, B)$  is defined as

$$\text{cap}^*(A, B) = \int_{\partial A} [\mathbb{M}^\dagger(x) \nabla h_{A,B}^*(x)] \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx) , \quad (3.4)$$

where  $h_{A,B}^* : \mathbb{R}^d \rightarrow [0, 1]$ , called the equilibrium potential between  $A$  and  $B$  for the adjoint generator, is the unique solution in  $C^2(\overline{\Omega}) \cap W^{1,2}(\Omega)$  of the elliptic equation

$$\begin{cases} (\mathcal{L}^* u)(x) = 0 & x \in \Omega , \\ u(x) = \chi_A(x) & x \in \partial \Omega . \end{cases}$$

The next lemma states that the capacity between two disjoint subsets  $A, B$  of  $\mathbb{R}^d$  coincides with the capacity with respect to the adjoint process. Recall that we are assuming that  $A$  and  $B$  fulfill the conditions S.

**Lemma 3.2.** *For every open subsets  $A, B$  of  $\mathbb{R}^d$ ,*

$$\text{cap}(A, B) = \text{cap}^*(A, B) .$$

*Proof.* As in the proof of Lemma 3.1, we may write  $\text{cap}(A, B)$  as

$$\begin{aligned} & \int_{\partial A} h_{A,B}^*(x) [\mathbb{M}(x) \nabla h_{A,B}(x)] \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx) \\ & - \int_{\partial B} h_{A,B}^*(x) [\mathbb{M}(x) \nabla h_{A,B}(x)] \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx) \\ & - \int_{\Omega} h_{A,B}^*(x) \nabla \cdot [e^{-V(x)} \mathbb{M}(x) \nabla h_{A,B}(x)] dx, \end{aligned} \quad (3.5)$$

By the arguments presented in the proof of the previous lemma, and the divergence theorem, this expression is equal to

$$\int_{\Omega} \nabla h_{A,B}^*(x) \mathbb{M}(x) \nabla h_{A,B}(x) \mu(dx) = \int_{\Omega} \nabla h_{A,B}(x) \mathbb{M}^\dagger(x) \nabla h_{A,B}^*(x) \mu(dx).$$

By the divergence theorem once more, we obtain that this integral is equal to the sum (3.5), in which  $\nabla h_{A,B}$  and  $\nabla h_{A,B}^*$  are interchanged and  $\mathbb{M}(x)$  is replaced by  $\mathbb{M}^\dagger(x)$ . We may remove the function  $h_{A,B}^*$  in the first line because it is equal to 1 on  $\partial A$ . The second line vanishes because  $h_{A,B}^*$  is equal to 0 at  $\partial B$ , and the third line vanishes because  $h_{A,B}^*$  is harmonic on  $\Omega$ . This completes the proof of the lemma.  $\square$

Recall from (2.10) the definition of the vector fields  $\Psi_f$ ,  $\Phi_f$ ,  $\Phi_f^*$ . Note that for every function  $f, g$  in  $C^2(\overline{\Omega}) \cap W^{1,2}(\Omega)$ ,

$$\langle \Psi_f, \Psi_g \rangle = \int_{\Omega} (\nabla f)(x) \cdot \mathbb{S}(x) (\nabla g)(x) \mu(dx).$$

In particular, by Lemma 3.1,

$$\text{cap}(A, B) = \langle \Psi_{h_{A,B}}, \Psi_{h_{A,B}} \rangle. \quad (3.6)$$

On the other hand, for every function  $f, g$  in  $C^2(\overline{\Omega}) \cap W^{1,2}(\Omega)$ ,

$$\begin{aligned} \langle \Phi_f, \Psi_g \rangle &= \int_{\Omega} (\nabla g)(x) \cdot \mathbb{M}^\dagger(x) (\nabla f)(x) \mu(dx), \\ \langle \Phi_f^*, \Psi_g \rangle &= \int_{\Omega} (\nabla g)(x) \cdot \mathbb{M}(x) (\nabla f)(x) \mu(dx). \end{aligned} \quad (3.7)$$

*Proof of Proposition 2.1.* We first claim that for all  $f \in \mathcal{C}_{A,B}^{1,0}$ ,  $\varphi \in \mathcal{F}^{(0)}$ ,

$$\langle \Phi_f - \varphi, \Psi_{h_{A,B}} \rangle = \text{cap}(A, B). \quad (3.8)$$

Indeed, on the one hand, for any  $f \in \mathcal{C}_{A,B}^{1,0}$ ,  $\varphi \in \mathcal{F}^{(0)}$ , by definition of  $\Psi_f$ , by the divergence theorem, and since  $f$  is bounded and belongs to  $W^{1,2}(\Omega)$ , and since  $\int_{\Omega} \|\varphi\|^2 \exp\{V(x)\} dx$  is finite,  $\langle \Psi_f, \varphi \rangle$  is equal to

$$\int_{\Omega} \varphi(x) \cdot (\nabla f)(x) dx = \int_{\partial\Omega} f(x) \varphi(x) \cdot \mathbf{n}_\Omega(x) \sigma(dx) - \int_{\Omega} \nabla \cdot \varphi(x) f(x) dx.$$

The second integral on the right hand side vanishes because  $\varphi$  is divergence free in  $\Omega$ , while the first integral vanishes because  $f$  is constant on each set  $\partial A$ ,  $\partial B$  and the vector field  $\varphi$  belongs to  $\mathcal{F}^{(0)}$ .

Therefore, by (3.7),

$$\langle \Phi_f - \varphi, \Psi_{h_{A,B}} \rangle = \langle \Phi_f, \Psi_{h_{A,B}} \rangle = \int_{\Omega} (\nabla f)(x) \cdot \mathbb{M}(x) (\nabla h_{A,B})(x) e^{-V(x)} dx.$$

By the divergence theorem, the previous expression is equal to

$$- \int_{\Omega} f(x) (\mathcal{L}h_{A,B})(x) \mu(dx) + \int_{\partial\Omega} f(x) e^{-V(x)} \mathbb{M}(x) (\nabla h_{A,B})(x) \cdot \mathbf{n}_{\Omega}(x) \sigma(dx) .$$

The first integral vanishes because  $h_{A,B}$  is harmonic on  $\Omega$ . Since  $f$  belongs to  $\mathcal{C}_{A,B}^{1,0}$ , we may first restrict the second integral to  $\partial A$ , and then remove the function  $f$  to conclude that

$$\langle \Phi_f - \varphi, \Psi_{h_{A,B}} \rangle = \int_{\partial A} e^{-V(x)} \mathbb{M}(x) (\nabla h_{A,B})(x) \cdot \mathbf{n}_{\Omega}(x) \sigma(dx) ,$$

which proves claim (3.8) in view of (2.8).

By (3.8) and by Schwarz inequality (2.9), for every  $f \in \mathcal{C}_{A,B}^{1,0}$ ,  $\varphi \in \mathcal{F}^{(0)}$ ,

$$\text{cap}(A, B)^2 = \langle \Phi_f - \varphi, \Psi_{h_{A,B}} \rangle^2 \leq \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle \langle \Psi_{h_{A,B}}, \Psi_{h_{A,B}} \rangle .$$

By (3.6), the last term is equal to  $\text{cap}(A, B)$ , which proves that

$$\langle \Phi_f - \varphi, \Phi_f - \varphi \rangle \geq \text{cap}(A, B)$$

for all  $f \in \mathcal{C}_{A,B}^{1,0}$ ,  $\varphi \in \mathcal{F}^{(0)}$ .

To complete the proof of the proposition, it remains to show that  $\varphi = \Phi_f - \Psi_{h_{A,B}}$  belongs to  $\mathcal{F}^{(0)}$  for  $f = (1/2)(h_{A,B} + h_{A,B}^*)$ . This is indeed the case. Recall that  $h_{A,B}$ ,  $h_{A,B}^*$  are bounded and belong to  $C^2(\overline{\Omega}) \cap W^{1,2}(\Omega)$ . On the one hand, by definition of  $f$ , for every  $x \in \Omega$ ,

$$\nabla \cdot [\Phi_f - \Psi_{h_{A,B}}] = e^{-V(x)} (\mathcal{L}^* f)(x) - e^{-V(x)} (\mathcal{S} h_{A,B})(x) ,$$

where  $\mathcal{S} = (1/2)(\mathcal{L} + \mathcal{L}^*)$ . Since  $f = (1/2)(h_{A,B} + h_{A,B}^*)$  by definition of  $\mathcal{S}$ , this expression is equal to

$$\frac{1}{2} e^{-V(x)} (\mathcal{L}^* h_{A,B}^*)(x) - \frac{1}{2} e^{-V(x)} (\mathcal{L} h_{A,B})(x) = 0 .$$

On the other hand, by definition of  $f$

$$\begin{aligned} & \int_{\partial A} [\Phi_f(x) - \Psi_{h_{A,B}}(x)] \cdot \mathbf{n}_{\Omega}(x) \sigma(dx) \\ &= \frac{1}{2} \int_{\partial A} e^{-V(x)} \mathbb{M}^{\dagger}(x) (\nabla h_{A,B}^*)(x) \cdot \mathbf{n}_{\Omega}(x) \sigma(dx) \\ & \quad - \frac{1}{2} \int_{\partial A} e^{-V(x)} \mathbb{M}(x) (\nabla h_{A,B})(x) \cdot \mathbf{n}_{\Omega}(x) \sigma(dx) . \end{aligned}$$

By definition of the capacities and by Lemma 3.2, this expression is equal to  $(1/2)\{\text{cap}(A, B) - \text{cap}^*(A, B)\} = 0$ . As  $\Phi_f - \Psi_{h_{A,B}}$  is divergence free on  $\Omega$ , the same identity holds at  $\partial B$ , which concludes the proof of the proposition.  $\square$

*Proof of Proposition 2.2.* We claim that for every  $f \in \mathcal{C}_{A,B}^{0,0}$ ,  $\varphi \in \mathcal{F}^{(1)}$ ,

$$\langle \Phi_f - \varphi, \Psi_{h_{A,B}} \rangle = -1 . \quad (3.9)$$

Indeed,

$$\langle \Phi_f - \varphi, \Psi_{h_{A,B}} \rangle = \int_{\Omega} (\nabla f)(x) \cdot \mathbb{M}(x) (\nabla h_{A,B})(x) \mu(dx) - \int_{\Omega} \varphi(x) \cdot (\nabla h_{A,B})(x) dx .$$

By the divergence theorem, this expression is equal to

$$\begin{aligned} & - \int_{\Omega} f(x) (\mathcal{L}h_{A,B})(x) \mu(dx) + \int_{\partial\Omega} f(x) \mathbb{M}(x) (\nabla h_{A,B})(x) \cdot \mathbf{n}_{\Omega}(x) e^{-V(x)} \sigma(dx) \\ & + \int_{\Omega} (\nabla \cdot \varphi)(x) h_{A,B}(x) dx - \int_{\partial\Omega} h_{A,B}(x) \varphi(x) \cdot \mathbf{n}_{\Omega}(x) \sigma(dx). \end{aligned}$$

The integrals over  $\Omega$  vanish because  $h_{A,B}$  is harmonic and the vector field  $\varphi$  is divergence free. The second integral in the first line also vanishes because  $f$  belongs to  $\mathcal{C}_{A,B}^{0,0}$ . The last integral is equal to  $-1$  because  $\varphi$  belongs to  $\mathcal{F}^{(1)}$ . This proves claim (3.9).

Therefore, by Schwarz inequality,

$$1 = \langle \Phi_f - \varphi, \Psi_{h_{A,B}} \rangle^2 \leq \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle \langle \Psi_{h_{A,B}}, \Psi_{h_{A,B}} \rangle.$$

Since  $\langle \Psi_{h_{A,B}}, \Psi_{h_{A,B}} \rangle = \text{cap}(A, B)$ , it follows from the previous relation that

$$\text{cap}(A, B) \geq \sup_{f \in \mathcal{C}_{A,B}^{0,0}} \sup_{\varphi \in \mathcal{F}^{(1)}} \frac{1}{\langle \Phi_f - \varphi, \Phi_f - \varphi \rangle}.$$

To complete the proof of the proposition, it remains to check that  $\varphi = \Phi_f - \Psi_{g_{A,B}}$  belongs to  $\mathcal{F}^{(1)}$ , where  $g_{A,B} = h_{A,B}/\text{cap}(A, B)$ ,  $f = (1/2)(h_{A,B} - h_{A,B}^*)/\text{cap}(A, B)$ . Since for  $x \in \Omega$ ,

$$\begin{aligned} (\nabla \cdot \varphi)(x) &= \frac{1}{2 \text{cap}(A, B)} e^{-V(x)} \left\{ [\mathcal{L}^*(h_{A,B} - h_{A,B}^*)](x) - 2(\mathcal{S}h_{A,B})(x) \right\} \\ &= \frac{-1}{2 \text{cap}(A, B)} e^{-V(x)} \left\{ (\mathcal{L}^*h_{A,B}^*)(x) + (\mathcal{L}h_{A,B})(x) \right\}, \end{aligned}$$

the vector field  $\varphi$  is divergence free on  $\Omega$ . On the other hand, the integral of  $\varphi \cdot \mathbf{n}_{\Omega}$  over the set  $\partial A$  is equal to

$$\frac{-1}{2 \text{cap}(A, B)} \int_{\partial A} \left\{ \mathbb{M}(x) (\nabla h_{A,B})(x) + \mathbb{M}^\dagger(x) (\nabla h_{A,B}^*)(x) \right\} \cdot \mathbf{n}_{\Omega}(x) e^{-V(x)} \sigma(dx).$$

By definition (2.8) and by Lemma 3.2, this expression is equal to  $-1$ , which proves that  $\varphi$  belongs to  $\mathcal{F}^{(1)}$ . This completes the proof of the proposition.  $\square$

**3.2. The reversible case.** When the matrix  $\mathbb{M}$  is symmetric and the generator  $\mathcal{L}$  is symmetric in  $L^2(\mu)$ , the previous variational formulae are simplified and we recover the Dirichlet and the Thomson principle for reversible diffusions.

Fix two open subsets  $A, B$  of  $\mathbb{R}^d$ . On the one hand, since  $h_{A,B} = h_{A,B}^*$ , and since all vector fields  $\Phi_f, \Phi_f^*, \Psi_f$  coincide, by Proposition 2.1, the minimum over  $\varphi$  in (2.11) is attained at  $\varphi = 0$ , so that

$$\text{cap}(A, B) = \inf_{f \in \mathcal{C}_{A,B}^{1,0}} \langle \Psi_f, \Psi_f \rangle = \inf_{f \in \mathcal{C}_{A,B}^{1,0}} \int_{\Omega} (\nabla f)(x) \cdot \mathbb{S}(x) (\nabla f)(x) \mu(dx), \quad (3.10)$$

which is the well-known Dirichlet principle.

Similarly, the supremum over  $f$  in (2.12) is attained at  $f = 0$ , so that

$$\text{cap}(A, B) = \sup_{\varphi \in \mathcal{F}^{(1)}} \frac{1}{\langle \varphi, \varphi \rangle}, \quad (3.11)$$

which is the classical Thomson principle.

**3.3. The harmonic measure.** Recall that the sets  $A, B$  are assumed to fulfill conditions S. Let  $\nu_{A,B}$  be the harmonic measure on  $\partial A$ :

$$\nu_{A,B}(dx) = \frac{1}{\text{cap}(A, B)} \mathbb{M}^\dagger(x) (\nabla h_{A,B}^*)(x) \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx). \quad (3.12)$$

Since  $\partial A$  is the 1-level set of the equilibrium potential  $h_{A,B}^*$  which, by the maximum principle, is bounded by 1,  $\nabla h_{A,B}^* = c \mathbf{n}_\Omega(x)$  for some  $c \geq 0$  so that  $\mathbb{M}^\dagger(x) (\nabla h_{A,B}^*)(x) \cdot \mathbf{n}_\Omega(x) = \mathbb{S}(x) (\nabla h_{A,B}^*)(x) \cdot \mathbf{n}_\Omega(x) \geq 0$ . This shows that  $\nu_{A,B}$  is a probability measure.

**Proposition 3.3.** *For any two bounded, open subsets  $A, B$  satisfying assumptions S, and for every bounded function  $f$  in  $C^\alpha(\mathbb{R}^d)$ ,*

$$\mathbb{E}_{\nu_{A,B}} \left[ \int_0^{H_B} f(X_s) ds \right] = \frac{1}{\text{cap}(A, B)} \int_{\mathbb{R}^d} h_{A,B}^*(x) f(x) e^{-V(x)} dx. \quad (3.13)$$

*Proof.* Fix a bounded function  $f$  in  $C^\alpha(\mathbb{R}^d)$ , and let  $\Omega_B = \mathbb{R}^d \setminus \overline{B}$ . Denote by  $u$  the unique solution in  $W^{1,2}(\Omega_B) \cap C^{2,\alpha}(\Omega_B)$  of the elliptic equation (2.6) with  $\Omega = \Omega_B$ ,  $\mathbf{g} = f$ ,  $\mathbf{b} = 0$ . In view of the representation (3.1) of  $u$  and by definition of the harmonic measure  $\nu_{A,B}$ , the left hand side of (3.13) is equal to

$$\frac{1}{\text{cap}(A, B)} \int_{\partial A} u(x) [\mathbb{M}^\dagger(x) \nabla h_{A,B}^*(x)] \cdot \mathbf{n}_\Omega(x) e^{-V(x)} \sigma(dx).$$

The integral of the same expression at  $\partial B$  vanishes due to the presence of the function  $u$ . Hence, by the divergence theorem, this expression is equal to

$$\frac{1}{\text{cap}(A, B)} \int_{\Omega} \nabla \cdot \left\{ [\mathbb{M}^\dagger(x) \nabla h_{A,B}^*(x)] e^{-V(x)} u(x) \right\} dx.$$

Since the equilibrium potential  $h_{A,B}^*$  is harmonic on  $\Omega$ , the previous equation is equal to

$$\frac{1}{\text{cap}(A, B)} \int_{\Omega} \nabla h_{A,B}^*(x) e^{-V(x)} \mathbb{M}(x) (\nabla u)(x) dx.$$

By the divergence theorem and since the equilibrium potential  $h_{A,B}^*$  is equal to 1 on  $\partial A$  and 0 on  $\partial B$ , this expression becomes

$$\begin{aligned} & \frac{1}{\text{cap}(A, B)} \int_{\partial A} e^{-V(x)} \mathbb{M}(x) (\nabla u)(x) \cdot \mathbf{n}_\Omega(x) \sigma(dx) \\ & - \frac{1}{\text{cap}(A, B)} \int_{\Omega} h_{A,B}^*(x) \nabla \cdot \left\{ e^{-V(x)} \mathbb{M}(x) (\nabla u)(x) \right\} dx. \end{aligned}$$

As  $\mathbf{n}_\Omega = -\mathbf{n}_A$  on  $\partial A$  and  $\mathcal{L}u = -f$  on  $A$ , the first integral vanishes is equal to

$$\frac{-1}{\text{cap}(A, B)} \int_A \nabla \cdot \left\{ e^{-V(x)} \mathbb{M}(x) (\nabla u)(x) \right\} dx = \frac{1}{\text{cap}(A, B)} \int_A f(x) e^{-V(x)} dx.$$

Since the equilibrium potential  $h_{A,B}^*$  is equal to 1 on  $A$ , we may insert it in the integral. On the other hand, by definition of  $u$ ,  $\nabla \cdot \{e^{-V(x)} \mathbb{M}(x) (\nabla u)(x)\} = \exp\{-V(x)\}(\mathcal{L}u)(x) = -\exp\{-V(x)\}f(x)$ . The second term of the penultimate displayed equation is thus equal to

$$\frac{1}{\text{cap}(A, B)} \int_{\Omega} h_{A,B}^*(x) f(x) e^{-V(x)} dx.$$

Since the equilibrium potential  $h_{A,B}^*$  vanishes at  $B$ , we may extend this integral to the set  $B$ , which completes the proof of the proposition.  $\square$

Proposition 3.3 can be restated as follows. Let  $u$  be the solution of (2.6) with  $\mathfrak{g} = -f$ ,  $\mathfrak{b} = 0$ ,  $\Omega = \overline{B}^c$ . Then,

$$\int_{\partial A} u(x) \nu_{A,B}(dx) = \frac{1}{\text{cap}(A, B)} \int_{\mathbb{R}^d} h_{A,B}^*(x) f(x) e^{-V(x)} dx. \quad (3.14)$$

#### 4. DIRICHLET AND THOMSON PRINCIPLES ON A COMPACT MANIFOLD

**4.1. Notation.** Let  $\mathfrak{M}$  be a compact manifold without boundary, equipped with a Riemannian tensor  $g = a^{-1}$ . We assume that the matrix  $a$  satisfies the ellipticity condition (2.2). Denote by  $\nabla$  the gradient, by  $\nabla \cdot$  the divergence and by  $\Delta$  the Laplace operators on  $\mathfrak{M}$ . The tangent and cotangent norms induced by the tensor  $g$  of a tangent or cotangent vector  $\eta$  are denoted by  $|\eta|$ , and the tangent-cotangent duality is simply denoted by  $\cdot$ . Thus, for instance,  $|\nabla V|^2$  stands for what has been denoted by  $a \nabla V \cdot \nabla V$  in the previous section.

Recall from [2, Definition 3.35, page 143] that a set  $A \subset \mathfrak{M}$  has finite perimeter if the indicator function of  $A$ , denoted by  $\chi_A$ , has bounded variation. In such a case the notation  $\mathbf{n}_A := \nabla \chi_A$  is used, so that  $\mathbf{n}_A$  represents the inward pointing unit normal field of the boundary  $\partial A$ . The volume measure on  $\mathfrak{M}$  is denoted by  $dx$ . If  $\mu(dx) = \varrho(x) dx$  for some continuous function  $\varrho : \mathfrak{M} \rightarrow \mathbb{R}$  and if  $A$  has finite perimeter, with some abuse of notation, we denote by  $\mu_A$  the measure  $\varrho(x) \sigma(dx)$  on  $\partial A$ . Hence, for every smooth tangent vector field  $\varphi$ ,

$$\oint \varphi(x) \cdot \mathbf{n}(x) \mu_A(dx) = \oint \varphi(x) \cdot \mathbf{n}(x) \varrho(x) \sigma(dx).$$

**4.2. Generator.** Denote by  $\tilde{\mathcal{L}}$  the generator given by

$$\tilde{\mathcal{L}}f = \Delta f + b \cdot \nabla f \quad f \in C^2(\mathfrak{M}),$$

where  $b : \mathfrak{M} \rightarrow \mathbb{R}^d$  is a smooth vector field. Since  $\mathfrak{M}$  is compact and the process is absolutely continuous with respect to the Wiener measure, there exists a unique Borel probability measure such that  $\mu \tilde{\mathcal{L}} = 0$ . Moreover,  $\mu(dx) = e^{-V(x)} dx$ , where  $V$  is the unique viscosity solution to

$$|\nabla V|^2 + b \cdot \nabla V = \Delta V + \nabla \cdot b.$$

Since  $a$  satisfies condition (2.2) and  $\varrho(x) = e^{-V(x)}$  is the solution of a linear second-order elliptic equation, by [12, Theorem 8.3],  $V$  is smooth.

The generator  $\tilde{\mathcal{L}}$  extends to a closed, unbounded operator  $\mathcal{L}$  on  $L_2(\mu)$ . It is easy to check that  $\mathcal{L}$  writes uniquely in the form

$$\mathcal{L}f = e^V \nabla \cdot (e^{-V} \nabla f) + c \cdot \nabla f \quad (4.1)$$

for a suitable vector field  $c$ , which is also smooth and satisfies  $\nabla \cdot (e^{-V} c) = 0$ . This in turn implies that for any  $A \subset \mathfrak{M}$ , and smooth functions  $f, g : \mathfrak{M} \rightarrow \mathbb{R}$ ,

$$\begin{aligned} \oint c \cdot \mathbf{n} d\mu_A &= \int_{\mathfrak{M} \setminus A} e^V \nabla \cdot (e^{-V} c) d\mu = 0, \\ \int f c \cdot \nabla g d\mu &= - \int g c \cdot \nabla f d\mu. \end{aligned} \quad (4.2)$$

Namely, the operator  $c \cdot \nabla$  is skew-adjoint in  $L_2(\mu)$ .

Denote by  $H^1 = H^1(\mathfrak{M})$  the Hilbert space of weakly differentiable functions endowed with the scalar product  $\langle f, g \rangle_1$  given by

$$\langle f, g \rangle_1 = \int f g d\mu + \int \nabla f \cdot \nabla g d\mu.$$

Functions in  $H^1$  admit a trace at the boundary of sets of finite perimeter. By [9, Theorem 2.1], the usual integration by parts formulae hold w.r.t. to this trace.

Let  $A$  and  $B$  be disjoint closed subsets of  $\mathfrak{M}$  with a finite perimeter and let  $f, g \in H^1$ . If  $f$  and  $g$  are such that  $f|_{\partial A}$ ,  $g|_{\partial A}$ ,  $f|_{\partial B}$  and  $g|_{\partial B}$  are (possibly different) constant, then

$$\int_{\mathfrak{M} \setminus A \cup B} f c \cdot \nabla g d\mu = - \int_{\mathfrak{M} \setminus A \cup B} g c \cdot \nabla f d\mu \quad (4.3)$$

since all the boundary terms in the integration by parts vanish in view of (4.2). In particular  $c \cdot \nabla$  is skew-adjoint on  $L_2(\mu|_{A \cup B})$  when restricted to  $H^1$  functions that take a constant value at the boundary.

On the other hand  $e^V \nabla \cdot (e^{-V} \nabla)$  is self-adjoint in  $L_2(\mu)$ , so that (4.1) provides a decomposition  $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_a$  in the symmetric and skew part of  $\mathcal{L}$  in  $L_2(\mu)$ . The adjoint  $\mathcal{L}^*$  of  $\mathcal{L}$  is then defined as

$$\mathcal{L}^* f = \mathcal{L}_s f - \mathcal{L}_a f = e^V \nabla \cdot (e^{-V} \nabla f) - c \cdot \nabla f. \quad (4.4)$$

**4.3. Stochastic processes.**  $\mathcal{L}$  and  $\mathcal{L}^*$  are the generators of a Feller process on  $\mathfrak{M}$  with invariant measure  $\mu$ . We denote by  $(\mathbb{P}_x)$  and  $(\mathbb{P}_x^*)$  the induced probability measures on  $C([0, +\infty); \mathfrak{M})$ .

If  $A$  is closed, let  $H_A$  be the hitting time of  $A$  and for a given  $f \in L_2(\mu)$  consider the function

$$u(x) := \mathbb{E}_x \left[ \int_0^{H_A} f(X_t) dt \right]. \quad (4.5)$$

If  $A$  is the closure of an open set with smooth boundary then  $u$  is the unique  $\mathcal{H}^1(\mathfrak{M} \setminus A)$  solution to

$$\begin{cases} \mathcal{L}u = f & \text{on } \mathfrak{M} \setminus A, \\ u = 0 & \text{on } \partial A. \end{cases} \quad (4.6)$$

Similarly, if  $A$  and  $B$  are closed, disjoint sets that are the closure of open sets with smooth boundary, define

$$h_{A,B}(x) = \mathbb{P}_x(H_A < H_B), \quad h_{A,B}^*(x) = \mathbb{P}_x^*(H_A < H_B).$$

Then  $h$  and  $h^*$  are the unique  $\mathcal{H}^1$  solutions to

$$\begin{cases} \mathcal{L}h = 0 & \text{on } \mathfrak{M} \setminus A \cup B, \\ h = 1 & \text{on } A, \\ h = 0 & \text{on } B, \end{cases} \quad \begin{cases} \mathcal{L}^*h^* = 0 & \text{on } \mathfrak{M} \setminus A \cup B, \\ h^* = 1 & \text{on } A, \\ h^* = 0 & \text{on } B. \end{cases} \quad (4.7)$$

**4.4. Capacity.** For  $\mathcal{L}$ ,  $A$  and  $B$  as above, the *capacities*  $\text{cap}(A, B)$ ,  $\text{cap}^*(A, B)$  are defined as

$$\text{cap}(A, B) := \int |\nabla h_{A,B}|^2 d\mu, \quad \text{cap}^*(A, B) := \int |\nabla h_{A,B}^*|^2 d\mu. \quad (4.8)$$

Hereafter we fix the sets  $A$  and  $B$  and denote  $h \equiv h_{A,B}$  and  $h^* \equiv h_{A,B}^*$ .



**Lemma 4.1.** *We have that*

$$\text{cap}(A, B) = \text{cap}(B, A) = \oint_{\partial A} (\nabla h) \cdot \mathbf{n} d\mu_A = \oint_{\partial A} (\nabla h + h c) \cdot \mathbf{n} d\mu_A .$$

Moreover,

$$\text{cap}(A, B) = \int \{ \nabla h \cdot \nabla h^* - h^* c \cdot \nabla h \} d\mu = \int \{ \nabla h \cdot \nabla h^* + h c \cdot \nabla h^* \} d\mu , \quad (4.9)$$

and  $\text{cap}(A, B) = \text{cap}^*(A, B)$ .

*Proof.*  $\text{cap}(B, A) = \text{cap}(A, B)$  since  $h_{B,A} = 1 - h_{A,B}$  as  $A \cap B = \emptyset$ . On the other hand, since  $h = \mathbf{1}\{\partial A\}$  on  $\partial A \cup \partial B$ , by the explicit form of the generator  $\mathcal{L}$  and by an integration by parts,

$$\begin{aligned} \int |\nabla h|^2 d\mu &= - \int h e^V \nabla \cdot (e^{-V} \nabla h) d\mu + \oint h \nabla h \cdot \mathbf{n} d\mu_{A \cup B} \\ &= \int h (c \cdot \nabla h) d\mu + \oint \nabla h \cdot \mathbf{n} d\mu_A . \end{aligned} \quad (4.10)$$

The first term vanishes in view of the second identity of (4.2). This proves the second assertion of the lemma. The third assertion follows from the first equation in (4.2) and from the fact that  $h$  is constant in  $\partial A$ .

A similar reasoning to the one in (4.10) yields

$$\int \nabla h \cdot \nabla h^* d\mu = \int h^* c \cdot \nabla h d\mu + \oint \nabla h \cdot \mathbf{n} d\mu_A = \int h^* c \cdot \nabla h d\mu + \text{cap}(A, B) ,$$

where the last identity follows from the first part of the proof. The previous equation is the first identity in (4.9). The second identity in (4.9) is obtained from (4.2). The same computations inverting the roles of  $h$  and  $h^*$  gives that

$$\int \nabla h \cdot \nabla h^* d\mu = - \int h c \cdot \nabla h^* d\mu + \text{cap}^*(A, B) .$$

In particular,  $\text{cap}(A, B) = \text{cap}^*(A, B)$ , which completes the proof of the lemma.  $\square$

Considering  $\mathcal{L}^*$  in place of  $\mathcal{L}$  we obtain from the previous lemma that

$$\text{cap}^*(A, B) = \text{cap}^*(B, A) = \oint_{\partial A} (\nabla h^*) \cdot \mathbf{n} d\mu_A = \oint_{\partial A} (\nabla h^* - h^* c) \cdot \mathbf{n} d\mu_A \quad (4.11)$$

**4.5. Harmonic measure.** Fix  $A$  and  $B$  as above. Define the probability measure  $\nu \equiv \nu_{A,B}$  as the harmonic measure on  $\partial A \cup \partial B$  conditioned to  $\partial A$  as

$$d\nu := \frac{-1}{\text{cap}(A, B)} \nabla h^* \cdot \mathbf{n} d\mu_A .$$

**Proposition 4.2.** *For each  $f \in L_2(\mu)$  it holds*

$$\mathbb{E}_\nu \left[ \int_0^{H_B} f(X_t) dt \right] = \frac{1}{\text{cap}(A, B)} \int h^* f d\mu .$$

*Proof.* Take  $u$  as in (4.5) with  $A$  changed to  $B$ . Since  $u$  vanishes on  $\partial B$ ,

$$\begin{aligned} \oint u \nabla h^* \cdot \mathbf{n} d\mu_A(x) &= \int_{\mathfrak{M} \setminus A \cup B} e^V \nabla \cdot (u e^{-V} \nabla h^*) d\mu \\ &= \int_{\mathfrak{M} \setminus A \cup B} \{ \nabla h^* \cdot \nabla u + u e^V \nabla \cdot (e^{-V} \nabla h^*) \} d\mu \quad (4.12) \\ &= \int_{\mathfrak{M} \setminus A \cup B} (\nabla h^* \cdot \nabla u + u c \cdot \nabla h^*) d\mu \end{aligned}$$

where we used the fact that  $\mathcal{L}^* h^* = 0$  in the last equality. Since  $\nabla h^*$  vanishes on  $A$ , the quantity in (4.12) also equals

$$\int_{\mathfrak{M} \setminus B} (\nabla h^* \cdot \nabla u + u c \cdot \nabla h^*) d\mu = \int_{\mathfrak{M} \setminus B} \{ -h^* e^V \nabla \cdot (e^{-V} \nabla u) + u c \cdot \nabla h^* \} d\mu$$

where in the last equality we used the fact that  $h^* = 0$  on  $\partial B$ , so that boundary terms vanish in the integration by parts. Since  $u$  satisfies (4.6) we gather

$$\oint u \nabla h^* \cdot \mathbf{n} d\mu_A = \int_{\mathfrak{M} \setminus B} \{ -h^* f + h^* c \cdot \nabla u + u c \cdot \nabla h^* \} d\mu.$$

However the last two terms sum up to zero since  $h$  and  $u$  are constant on  $\partial B$  and (4.3) holds (with  $A = \emptyset$ ). Thus, since  $h^*$  vanishes on  $B$ ,

$$\oint u \nabla h^* \cdot \mathbf{n} d\mu_A = - \int_{\mathfrak{M}} h^* f d\mu.$$

Therefore, by linearity of the expectation and by the previous equation,

$$\begin{aligned} \mathbb{E}_\nu \left[ \int_0^{H_B} f(X_t) dt \right] &= \oint u d\nu = \frac{-1}{\text{cap}(A, B)} \oint u \nabla h^* \cdot \mathbf{n} d\mu_A \\ &= \frac{1}{\text{cap}(A, B)} \int h^* f d\mu. \end{aligned}$$

□

**4.6. Variational formulae for the capacity.** In view of Proposition 4.2, it may be useful to have variational formulae for the capacity in order to estimate the expected value of hitting times.

Let  $\mathcal{F} \equiv \mathcal{F}_{A,B}$  be the Hilbert space of  $L_2(\mu|_{\mathfrak{M} \setminus A \cup B})$  tangent vector fields on  $\mathfrak{M} \setminus A \cup B$ , and let  $\langle \cdot, \cdot \rangle$  be the associated scalar product:

$$\langle \varphi, \psi \rangle := \int_{\mathfrak{M} \setminus A \cup B} a^{-1} \varphi \cdot \psi d\mu.$$

For  $\gamma \in \mathbb{R}$  let also  $\mathcal{F}^\gamma$  be the closure in  $\mathcal{F}$  of the space of smooth tangent vector fields  $\varphi \in \mathcal{F}$  such that

$$\nabla \cdot (e^{-V} \varphi) = 0, \quad \oint \varphi \cdot \mathbf{n} d\mu_A = -\gamma. \quad (4.13)$$

It is a well-known fact that  $\mathcal{F}^\gamma$  is the space of tangent vector fields such that  $\nabla \cdot (e^{-V} \varphi) = 0$  weakly, and that such vector fields admit a weak normal trace  $\varphi \cdot \mathbf{n}$  such that (4.13) holds.

Let also  $\mathcal{H}_{\alpha,\beta} \equiv \mathcal{H}_{\alpha,\beta,A,B}$  be the space of  $H^1$  functions  $f$  on  $\mathfrak{M} \setminus A \cup B$  such that their normal trace at  $A$  and  $B$  is constant and equal to  $\alpha$  and  $\beta$  respectively (these

traces exist since we assumed  $A$  and  $B$  to have finite perimeter). For  $f \in \mathcal{H}_{\alpha,\beta}$  define  $\Phi_f := \nabla f - c f$ .

**Lemma 4.3.** *If  $\varphi \in \mathcal{F}^\gamma$  and  $f \in \mathcal{H}_{\alpha,0}$  then*

$$\langle \Phi_f - \varphi, \nabla h \rangle = \gamma + \alpha \operatorname{cap}(A, B).$$

*Proof.* By definition of  $\Phi_f$ ,

$$\langle \Phi_f - \varphi, \nabla h \rangle = \int_{\mathfrak{M} \setminus A \cup B} (\nabla f - f c - \varphi) \cdot \nabla h \, d\mu.$$

Integrating by parts, since  $f = \alpha$  on  $\partial A$  and  $f = 0$  on  $\partial B$ , the previous term becomes

$$- \int_{\mathfrak{M} \setminus A \cup B} \left\{ f e^V \nabla \cdot (e^{-V} \nabla h) + [f c + \varphi] \cdot \nabla h \right\} d\mu + \alpha \oint \nabla h \cdot \mathbf{n} \, d\mu_A.$$

By Lemma 4.1, the last integral is the capacity between  $A$  and  $B$ , while the expression involving  $f$  is equal to  $-f \mathcal{L}h$ . Since  $h$  is  $\mathcal{L}$ -harmonic in  $\mathfrak{M} \setminus A \cup B$ , by an integration by part, the previous equation is equal

$$\int_{\mathfrak{M} \setminus A \cup B} h e^V \nabla \cdot (e^{-V} \varphi) \, d\mu - \oint \varphi \cdot \mathbf{n} \, d\mu_A + \alpha \operatorname{cap}(A, B).$$

By (4.13), this expression is equal to  $\gamma + \alpha \operatorname{cap}(A, B)$ , as claimed.  $\square$

**Proposition 4.4** (Dirichlet principle). *It holds*

$$\operatorname{cap}(A, B) = \inf_{f \in \mathcal{H}_{1,0}} \inf_{\varphi \in \mathcal{F}^0} \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle, \quad (4.14)$$

and the infimum is attained for  $\bar{f} = (1/2)(h + h^*)$  and  $\bar{\varphi} = \Phi_{\bar{f}} - \nabla h$ .

*Proof.* From Lemma 4.3 (applied with  $\gamma = 0$  and  $\alpha = 1$ ), for  $f$  and  $\varphi$  as in (4.14), by Schwarz inequality,

$$\begin{aligned} \operatorname{cap}(A, B)^2 &= \langle \Phi_f - \varphi, \nabla h \rangle^2 \leq \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle \langle \nabla h, \nabla h \rangle \\ &= \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle \operatorname{cap}(A, B) \end{aligned}$$

so that  $\operatorname{cap}(A, B) \leq \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle$  for every  $f$  and  $\varphi$  as in (4.14).

Since  $\operatorname{cap}(A, B) = \langle \Phi_{\bar{f}} - \bar{\varphi}, \Phi_{\bar{f}} - \bar{\varphi} \rangle$ , to complete the proof of the proposition, one only needs to check that  $\bar{f} \in \mathcal{H}_{1,0}$ , and  $\bar{\varphi} \in \mathcal{F}^0$ . It is easy to check the first condition, while the second one follows from the identities

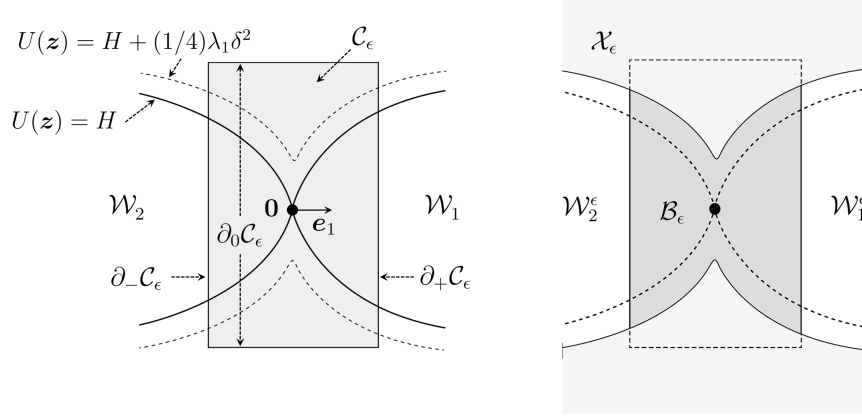
$$\begin{aligned} \nabla \cdot (e^{-V} \bar{\varphi}) &= (1/2) e^{-V} (\mathcal{L}^* h^* - \mathcal{L}h) = 0, \\ \oint \bar{\varphi} \cdot \mathbf{n} \, d\mu_A &= \frac{1}{2} \oint (\nabla h^* - h^* c) \cdot \mathbf{n} \, d\mu_A - \frac{1}{2} \oint (\nabla h + h c) \cdot \mathbf{n} \, d\mu_A. \end{aligned}$$

By Lemma 4.1 and (4.11), the previous expression is equal to  $(1/2)\{\operatorname{cap}^*(A, B) - \operatorname{cap}(A, B)\} = 0$ , which completes the proof of the proposition.  $\square$

**Proposition 4.5** (Thompson principle). *It holds*

$$\operatorname{cap}(A, B) = \sup_{f \in \mathcal{H}_{0,0}} \sup_{\varphi \in \mathcal{F}^1} \frac{1}{\langle \Phi_f - \varphi, \Phi_f - \varphi \rangle}. \quad (4.15)$$

Moreover, the supremum is attained at  $\bar{f} = (h - h^*)/2 \operatorname{cap}(A, B)$  and  $\bar{\varphi} = \Phi_{\bar{f}} - \nabla h / \operatorname{cap}(A, B)$ .

FIGURE 2. The neighborhood of the saddle point  $\mathbf{0}$ .

*Proof.* By Lemma 4.3 (applied with  $\alpha = 0$  and  $\gamma = 1$ ) and by Schwarz inequality, for  $f$  and  $\varphi$  as in (4.15) we have that

$$1 = \langle \Phi_f - \varphi, \nabla h \rangle^2 \leq \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle \langle \nabla h, \nabla h \rangle = \langle \Phi_f - \varphi, \Phi_f - \varphi \rangle \text{cap}(A, B).$$

Since  $\langle \Phi_{\bar{f}} - \bar{\varphi}, \Phi_{\bar{f}} - \bar{\varphi} \rangle = 1/\text{cap}(A, B)$ , one only need to check that  $\bar{f} \in \mathcal{H}_{0,0}$ , and  $\bar{\varphi} \in \mathcal{F}^1$ . It is easy to verify the first condition, while the second follows from

$$\begin{aligned} \nabla \cdot (e^{-V} \bar{\varphi}) &= \frac{-1}{2 \text{cap}(A, B)} e^{-V} (\mathcal{L}^* h^* + \mathcal{L} h) = 0 \\ \oint \bar{\varphi} \cdot \mathbf{n} d\mu_A &= \frac{-1}{2 \text{cap}(A, B)} \left\{ \oint (\nabla h^* - h^* c) \cdot \mathbf{n} d\mu_A + \oint (\nabla h + hc) \cdot \mathbf{n} d\mu_A \right\}. \end{aligned}$$

By Lemma 4.1 and (4.11), the expression inside braces is equal to  $\text{cap}^*(A, B) + \text{cap}(A, B) = 2 \text{cap}(A, B)$ , so that  $\bar{\varphi} \in \mathcal{F}^1$ . This completes the proof of the proposition.  $\square$

## 5. PROOF OF THEOREM 2.3

Throughout this section, to avoid unnecessary technical considerations, we assume that there is a unique saddle point of height  $H$  between the two valleys around  $\mathbf{m}_1$  and  $\mathbf{m}_2$ :  $\mathfrak{S} = \{\sigma\}$ . The general case can be handled without much effort. We refer to [17] for the details.

By a translation and change of coordinates we may assume that  $\sigma = \mathbf{0}$  and  $(\text{Hess } U)(\mathbf{0}) = \mathbb{L}^0 = \text{diag}(-\lambda_1^0, \lambda_2^0, \dots, \lambda_d^0)$ . We shall drop the superscript  $\mathbf{0}$  in these notations from now on. According to these assumptions, the eigenvectors of  $\mathbb{L}$  are the vectors of the canonical basis of  $\mathbb{R}^d$ , represented by  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ . Assume, furthermore, that  $\mathbf{e}_1$  is directed toward  $\mathcal{W}_1$ , i.e., that there exists  $t_0 > 0$  such that  $t\mathbf{e}_1 \in \mathcal{W}_1$  for all  $t \in (0, t_0)$ . (cf. Figure 2)

**A neighborhood of the saddle point.** We first introduce a neighborhood of the saddle point, cf. [17]. For a large enough constant  $K$  define

$$\delta := K \sqrt{\epsilon \log(1/\epsilon)}. \quad (5.1)$$

Let  $\mathcal{C}_\epsilon$  be the closed hyperrectangle around the saddle point  $\mathbf{0}$  defined by

$$\mathcal{C}_\epsilon = [-\delta, \delta] \times \prod_{i=2}^d \left[ -\sqrt{\frac{2\lambda_1}{\lambda_i}} \delta, \sqrt{\frac{2\lambda_1}{\lambda_i}} \delta \right],$$

and denote by  $\partial\mathcal{C}_\epsilon$  its boundary. Write  $\mathbf{z} \in \mathbb{R}^d$  as  $\mathbf{z} = \sum_{i=1}^d z_i \mathbf{e}_i$ , and define the boundaries  $\partial_-\mathcal{C}_\epsilon$ ,  $\partial_+\mathcal{C}_\epsilon$ ,  $\partial_0\mathcal{C}_\epsilon$  by

$$\begin{aligned} \partial_+\mathcal{C}_\epsilon &= \{\mathbf{z} \in \partial\mathcal{C}_\epsilon : z_1 = \delta\}, \quad \partial_-\mathcal{C}_\epsilon = \{\mathbf{z} \in \partial\mathcal{C}_\epsilon : z_1 = -\delta\}, \\ \partial_0\mathcal{C}_\epsilon &= \partial\mathcal{C}_\epsilon \setminus (\partial_-\mathcal{C}_\epsilon \cup \partial_+\mathcal{C}_\epsilon). \end{aligned}$$

Recall that  $U(\mathbf{0}) = H$ .

**Lemma 5.1.** *For all  $\mathbf{z} \in \partial_0\mathcal{C}_\epsilon$ , we have that  $U(\mathbf{z}) \geq H + [1 + o_\epsilon(1)](1/2)\lambda_1\delta^2$ .*

*Proof.* For  $\mathbf{z} \in \mathcal{C}_\epsilon$ , by the Taylor expansion,

$$U(\mathbf{z}) = H - \frac{1}{2}\lambda_1 z_1^2 + \frac{1}{2} \sum_{j=2}^d \lambda_j z_j^2 + O(\delta^3).$$

For  $\mathbf{z} \in \partial_0\mathcal{C}_\epsilon$ , there exists  $2 \leq i \leq d$ , such that  $z_i = \pm\sqrt{2\lambda_1/\lambda_i}\delta$ . Therefore,

$$-\lambda_1 z_1^2 + \sum_{j=2}^d \lambda_j z_j^2 \geq -\lambda_1 \delta^2 + \lambda_i \left( \sqrt{\frac{2\lambda_1}{\lambda_i}} \delta \right)^2 = \lambda_1 \delta^2.$$

To complete the proof, it remains to report this estimate to the first identity.  $\square$

Let

$$\Omega_\epsilon = \{\mathbf{z} \in \mathbb{R}^d : U(\mathbf{z}) < H + (1/4)\lambda_1\delta^2\}, \quad \mathcal{B}_\epsilon = \mathcal{C}_\epsilon \cap \Omega_\epsilon.$$

Since the saddle point  $\sigma = \mathbf{0}$  is the unique critical point separating the two local minima  $\mathbf{m}_1, \mathbf{m}_2$  of  $U$ , by Lemma 5.1, for  $\epsilon$  small enough, there are two connected components  $\mathcal{W}_1^\epsilon$  and  $\mathcal{W}_2^\epsilon$  of  $\Omega_\epsilon \setminus \mathcal{B}_\epsilon$  such that  $\mathbf{m}_i \in \mathcal{W}_i^\epsilon$ ,  $i = 1, 2$ . Note that  $\mathcal{V}_i \subset \mathcal{W}_i^\epsilon$ ,  $i = 1, 2$ , for sufficiently small  $\epsilon$ . Let  $\mathcal{X}_\epsilon = \mathbb{R}^d \setminus (\mathcal{W}_1^\epsilon \cup \mathcal{W}_2^\epsilon \cup \mathcal{B}_\epsilon)$ . These sets are represented in Figure 2.

**Approximations of the equilibrium potentials.** We introduce in this subsection an approximation of the equilibrium potentials  $h_{\mathcal{V}_1, \mathcal{V}_2}$ ,  $h_{\mathcal{V}_1^*, \mathcal{V}_2^*}$ . As pointed out in [8] for reversible diffusions and in [17] for non-reversible Markov chains, the crucial point consists in defining these approximations in a mesoscopic neighborhood of the saddle point, denoted above by  $\mathcal{B}_\epsilon$ .

Let  $-\mu$  be the unique negative eigenvalue of the matrices  $\mathbb{L}\mathbb{M}$ ,  $\mathbb{L}\mathbb{M}^\dagger$ , and let  $\mathbf{v}, \mathbf{v}^*$  be the associated normal eigenvectors. By Lemma 7.1 below, the first component of  $\mathbf{v}$ , denoted by  $v_1$ , does not vanish. Assume, without loss of generality, that  $v_1 > 0$ . Similarly, assume that  $v_1^*$ , the first component of  $\mathbf{v}^*$ , is positive.

Let

$$\alpha = \frac{\mu}{\mathbf{v} \cdot \mathbb{M}\mathbf{v}}, \quad \alpha^* = \frac{\mu}{\mathbf{v}^* \cdot \mathbb{M}\mathbf{v}^*}, \quad (5.2)$$

and let

$$C_\epsilon = \int_{-\infty}^{\infty} \exp\left\{-\frac{\alpha}{2\epsilon}t^2\right\} dt, \quad C_\epsilon^* = \int_{-\infty}^{\infty} \exp\left\{-\frac{\alpha^*}{2\epsilon}t^2\right\} dt.$$

Of course,  $C_\epsilon = \sqrt{2\pi\epsilon/\alpha}$ ,  $C_\epsilon^* = \sqrt{2\pi\epsilon/\alpha^*}$ . The constants  $\alpha$  and  $\alpha^*$  were introduced in [17] and they play a significant role in the estimation of the capacity.

Since  $\nabla U(\mathbf{z}) = \mathbb{L}\mathbf{z} + O(\|\mathbf{z}\|^2)$ , denote by  $\tilde{\mathcal{L}}_\epsilon$  the approximation of the generator  $\mathcal{L}_\epsilon$  around the origin, namely,

$$(\tilde{\mathcal{L}}_\epsilon f)(\mathbf{z}) = -(\mathbb{L}\mathbf{z}) \cdot \mathbb{M}(\nabla f)(\mathbf{z}) + \epsilon \sum_{1 \leq i, j \leq d} \mathbb{M}_{ij}(\partial_{z_i, z_j}^2 f)(\mathbf{z}).$$

Since the equilibrium potential satisfy the boundary conditions  $f \simeq 1$  on  $\partial \mathcal{B}_\epsilon \cap \partial_- \mathcal{C}_\epsilon$  and  $f \simeq 0$  on  $\partial \mathcal{B}_\epsilon \cap \partial_+ \mathcal{C}_\epsilon$ , a natural approximation of the equilibrium potentials  $h_{\mathcal{V}_1, \mathcal{V}_2}$ ,  $h_{\mathcal{V}_1, \mathcal{V}_2}^*$  in the neighborhood  $\mathcal{B}_\epsilon$  are

$$\begin{cases} p_\epsilon(\mathbf{z}) = (1/C_\epsilon) \int_{-\infty}^{\mathbf{z} \cdot \mathbf{v}} \exp\{-(\alpha/2\epsilon)t^2\} dt & \text{for } \mathbf{z} \in \mathcal{B}_\epsilon, \\ p_\epsilon(\mathbf{z}) = \mathbf{1}\{\mathbf{z} \in \mathcal{W}_1^\epsilon\} & \text{for } \mathbf{z} \in \mathcal{B}_\epsilon^c. \end{cases} \quad (5.3)$$

$$\begin{cases} p_\epsilon^*(\mathbf{z}) = (1/C_\epsilon^*) \int_{-\infty}^{\mathbf{z} \cdot \mathbf{v}^*} \exp\{-(\alpha^*/2\epsilon)t^2\} dt & \text{for } \mathbf{z} \in \mathcal{B}_\epsilon, \\ p_\epsilon^*(\mathbf{z}) = \mathbf{1}\{\mathbf{z} \in \mathcal{W}_1^\epsilon\} & \text{for } \mathbf{z} \in \mathcal{B}_\epsilon^c. \end{cases}$$

Note that  $p_\epsilon(\mathbf{z}) = 1$  on  $\mathcal{W}_1^\epsilon$  and that  $p_\epsilon(\mathbf{z}) = 0$  on  $\mathcal{W}_2^\epsilon \cup \mathcal{X}_\epsilon$ , and that  $p_\epsilon^*$  satisfies the same identities. Moreover,  $p_\epsilon$  and  $p_\epsilon^*$  are smooth in the interior of  $\mathcal{B}_\epsilon$ ,  $\mathcal{W}_1^\epsilon$ ,  $\mathcal{W}_2^\epsilon$  and  $\mathcal{X}_\epsilon$ , but have jumps along the boundaries of these domains. These jumps should be removed in order to use these functions as test functions for the Dirichlet and the Thomson principles.

To introduce the vector fields  $\Theta_{\mathbf{q}_\epsilon}$ ,  $\Theta_{\mathbf{q}_\epsilon}^*$ ,  $\Theta_{\mathbf{q}_\epsilon^*}$  and  $\Theta_{\mathbf{q}_\epsilon^*}^*$  which approximate the vectors fields  $\Phi_{h_{\mathcal{V}_1, \mathcal{V}_2}}$ ,  $\Phi_{h_{\mathcal{V}_1, \mathcal{V}_2}}^*$ ,  $\Phi_{h_{\mathcal{V}_1, \mathcal{V}_2}^*}$  and  $\Phi_{h_{\mathcal{V}_1, \mathcal{V}_2}^*}^*$ , respectively, let

$$\mathbf{q}_\epsilon(\mathbf{z}) = \begin{cases} (\nabla p_\epsilon)(\mathbf{z}) & \text{if } \mathbf{z} \in \mathcal{B}_\epsilon \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{q}_\epsilon^*(\mathbf{z}) = \begin{cases} (\nabla p_\epsilon^*)(\mathbf{z}) & \text{if } \mathbf{z} \in \mathcal{B}_\epsilon \\ 0 & \text{otherwise;} \end{cases}$$

and set

$$\begin{aligned} \Theta_{\mathbf{q}_\epsilon}(\mathbf{z}) &= \frac{\epsilon}{Z_\epsilon} e^{-U(\mathbf{z})/\epsilon} \mathbb{M}^\dagger \mathbf{q}_\epsilon(\mathbf{z}), \quad \Theta_{\mathbf{q}_\epsilon}^*(\mathbf{z}) = \frac{\epsilon}{Z_\epsilon} e^{-U(\mathbf{z})/\epsilon} \mathbb{M} \mathbf{q}_\epsilon(\mathbf{z}), \\ \Theta_{\mathbf{q}_\epsilon^*}(\mathbf{z}) &= \frac{\epsilon}{Z_\epsilon} e^{-U(\mathbf{z})/\epsilon} \mathbb{M}^\dagger \mathbf{q}_\epsilon^*(\mathbf{z}), \quad \Theta_{\mathbf{q}_\epsilon^*}^*(\mathbf{z}) = \frac{\epsilon}{Z_\epsilon} e^{-U(\mathbf{z})/\epsilon} \mathbb{M} \mathbf{q}_\epsilon^*(\mathbf{z}). \end{aligned}$$

One could be tempted to set  $\Theta_{\mathbf{q}_\epsilon} = \Phi_{p_\epsilon}$ . One has to be cautious, however, because  $p_\epsilon$  is discontinuous along  $\partial \mathcal{B}_\epsilon$ , and these jumps become significant when applying the divergence theorem.

Let  $T_\epsilon$  be the time scale given by

$$T_\epsilon := \frac{1}{Z_\epsilon} e^{-H/\epsilon} \frac{(2\pi\epsilon)^{d/2}}{2\pi}. \quad (5.4)$$

In the presence of a unique saddle point separating two wells, Theorem 2.3 becomes

**Theorem 5.2.** *We have that*

$$\text{cap}(\mathcal{V}_1, \mathcal{V}_2) = [1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0}). \quad (5.5)$$

In view of the explicit expression for the minimizers of the variational problem (2.11) in Proposition 2.1, the function  $f_\epsilon = (1/2)(p_\epsilon + p_\epsilon^*)$  and the vector field  $\varphi_\epsilon = (1/2)(\Theta_{\mathbf{q}_\epsilon^*} - \Theta_{\mathbf{q}_\epsilon}^*)$  are the natural candidates to estimate the capacity (5.5) through (2.11). However,  $f_\epsilon$  does not belong to the set  $\mathcal{C}_{\mathcal{V}_1, \mathcal{V}_2}^{1,0}$ , being discontinuous along the  $(d-1)$ -dimensional surface  $\partial \mathcal{X}_\epsilon \cup \partial \mathcal{B}_\epsilon$ . To overcome this difficulty, we convolve  $f_\epsilon$  with a smooth mollifier  $\phi_\eta(\cdot) := (1/\eta^d)\phi(\cdot/\eta)$ , where  $\phi$  is supported on the  $d$ -dimensional unit ball.

Denote by  $g^{(\eta)}$  the convolution of a function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  with the mollifier  $\phi_\eta$ :  $g^{(\eta)} := g * \phi_\eta$ . It follows from the explicit expression of  $p_\epsilon$  and  $p_\epsilon^*$  that  $p_\epsilon^{(\eta)}$  and  $(p_\epsilon^*)^{(\eta)}$  belongs to the set  $\mathcal{C}_{\mathcal{V}_1, \mathcal{V}_2}^{1,0}$  for sufficiently small  $\eta$ .

We turn to the test vector field  $\varphi_\epsilon = (1/2)(\Theta_{\mathbf{q}_\epsilon^*} - \Theta_{\mathbf{q}_\epsilon}^*)$ . It violates the requirements of the variational formula (2.11) in two ways: it is discontinuous along  $\partial\mathcal{B}_\epsilon$ , and it is not divergence-free on  $\mathcal{B}_\epsilon$ . The mollification procedure solves the first problem, but is not helpful for the second one. For this reason, instead of trying to smooth  $\varphi_\epsilon$  and to apply Propositions 2.1, we insert this vector field directly in the proof of this proposition and we estimate the error terms coming from the lack of regularity of the vector field.

*Proof of Theorem 5.2.* We start with the upper bound. Let

$$f_\epsilon^{(\epsilon)} = \frac{1}{2} \left\{ p_\epsilon^{(\epsilon)} + (p_\epsilon^*)^{(\epsilon)} \right\}, \quad \varphi_\epsilon = \frac{1}{2} (\Theta_{\mathbf{q}_\epsilon^*} - \Theta_{\mathbf{q}_\epsilon}^*).$$

Although  $\varphi_\epsilon$  does not satisfy the hypotheses of Proposition 2.1, inserting  $\varphi_\epsilon$  in the proof of Proposition 2.1 provides an upper bound for the capacity.

By the Schwarz inequality,

$$\langle \Phi_{f_\epsilon^{(\epsilon)}} - \varphi_\epsilon, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle^2 \leq \| \Phi_{f_\epsilon^{(\epsilon)}} - \varphi_\epsilon \|^2 \| \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \|^2. \quad (5.6)$$

Since, for sufficiently small  $\epsilon$ ,  $f_\epsilon^{(\epsilon)}$  belongs to  $\mathcal{C}_{\mathcal{V}_1, \mathcal{V}_2}^{1,0}$ , by the proof of Proposition 2.1,

$$\langle \Phi_{f_\epsilon^{(\epsilon)}}, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle = \text{cap}(\mathcal{V}_1, \mathcal{V}_2).$$

Therefore, by Lemma 5.5,

$$\langle \Phi_{f_\epsilon^{(\epsilon)}} - \varphi_\epsilon, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle = \text{cap}(\mathcal{V}_1, \mathcal{V}_2) + o_\epsilon(1) T_\epsilon. \quad (5.7)$$

On the other hand, by (5.11) and by the triangle inequality,

$$\| \Phi_{f_\epsilon^{(\epsilon)}} - \varphi_\epsilon \| \leq \left\| \frac{\Theta_{\mathbf{q}_\epsilon} + \Theta_{\mathbf{q}_\epsilon^*}}{2} - \varphi_\epsilon \right\| + o_\epsilon(1) \sqrt{T_\epsilon}.$$

Since

$$\frac{\Theta_{\mathbf{q}_\epsilon} + \Theta_{\mathbf{q}_\epsilon^*}}{2} - \varphi_\epsilon = \frac{\Theta_{\mathbf{q}_\epsilon} + \Theta_{\mathbf{q}_\epsilon^*}}{2},$$

by the last two displayed equations and by Lemma 5.3,

$$\| \Phi_{f_\epsilon^{(\epsilon)}} - \varphi_\epsilon \|^2 \leq [1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0}). \quad (5.8)$$

By (5.6), (5.7), (5.8), and since  $\| \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \|^2 = \text{cap}(\mathcal{V}_1, \mathcal{V}_2)$ ,

$$\{ \text{cap}(\mathcal{V}_1, \mathcal{V}_2) + o_\epsilon(1) T_\epsilon \}^2 \leq [1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0}) \text{cap}(\mathcal{V}_1, \mathcal{V}_2).$$

so that,

$$\text{cap}(\mathcal{V}_1, \mathcal{V}_2) \leq [1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0}).$$

This is the upper bound for the capacity.

In order to obtain the lower bound, we repeat the proof of Proposition 2.2. Let

$$g_\epsilon^{(\epsilon)} = \frac{p_\epsilon^{(\epsilon)} - (p_\epsilon^*)^{(\epsilon)}}{2 T_\epsilon \omega(\mathbf{0})}, \quad \psi_\epsilon = - \frac{\Theta_{\mathbf{q}_\epsilon}^* + \Theta_{\mathbf{q}_\epsilon^*}}{2 T_\epsilon \omega(\mathbf{0})}.$$

By the Schwarz inequality,

$$\langle \Phi_{g_\epsilon^{(\epsilon)}} - \psi_\epsilon, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle^2 \leq \| \Phi_{g_\epsilon^{(\epsilon)}} - \psi_\epsilon \|^2 \| \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \|^2. \quad (5.9)$$

Since  $g_\epsilon \in \mathcal{C}_{\mathcal{V}_1, \mathcal{V}_2}^{0,0}$  for sufficiently small  $\epsilon$ , as in the proof of Proposition 2.2, we obtain that

$$\langle \Phi_{g_\epsilon^{(\epsilon)}}, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle = 0.$$

On the other hand, by Lemma 5.5,

$$\langle \psi_\epsilon, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle = 1 + o_\epsilon(1).$$

In particular, the left hand side of (5.9) is equal to  $1 + o_\epsilon(1)$ .

Consider the first term on the right hand side of (5.9). By (5.11) and by the triangle inequality,

$$\|\Phi_{g_\epsilon^{(\epsilon)}} - \psi_\epsilon\| \leq \left\| \frac{\Theta_{\mathbf{q}_\epsilon} - \Theta_{\mathbf{q}_\epsilon^*}}{2T_\epsilon \omega(\mathbf{0})} - \psi_\epsilon \right\| + \frac{o_\epsilon(1)}{\sqrt{T_\epsilon}}. \quad (5.10)$$

Since

$$\frac{\Theta_{\mathbf{q}_\epsilon} - \Theta_{\mathbf{q}_\epsilon^*}}{2T_\epsilon \omega(\mathbf{0})} - \psi_\epsilon = \frac{\Theta_{\mathbf{q}_\epsilon} + \Theta_{\mathbf{q}_\epsilon^*}}{2T_\epsilon \omega(\mathbf{0})},$$

by Lemma 5.3, the right hand side of (5.10) is less than or equal to  $[1 + o_\epsilon(1)]\{T_\epsilon \omega(\mathbf{0})\}^{-1/2}$ . Putting together the previous estimates, since  $\|\Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}}\|^2 = \text{cap}(\mathcal{V}_1, \mathcal{V}_2)$ , we obtain from (5.9) that

$$[1 + o_\epsilon(1)]^2 \leq [1 + o_\epsilon(1)] \frac{1}{T_\epsilon \omega(\mathbf{0})} \text{cap}(\mathcal{V}_1, \mathcal{V}_2).$$

This completes the proof of lower bound.  $\square$

We conclude this section with three lemmata, whose proofs are postponed to Section 7.

**Lemma 5.3.** *We have that*

$$\left\| \frac{\Theta_{\mathbf{q}_\epsilon} + \Theta_{\mathbf{q}_\epsilon^*}}{2} \right\|^2 = [1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0}).$$

Recall from (5.1) that  $\delta = K\sqrt{\epsilon \log(1/\epsilon)}$ , where  $K$  is an arbitrary positive number.

**Lemma 5.4.** *Assume that  $\eta \ll \delta$ , in the sense that  $\lim_{\epsilon \rightarrow 0} \eta(\epsilon)/\delta(\epsilon) = 0$ . There exist positive constants  $C_1, C_2, C_3$  and  $C_4$ , which do not depend on  $\epsilon$  and  $\eta$ , such that*

$$\begin{aligned} & \|\Phi_{p_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon}\|^2 \\ & \leq \frac{C_1}{Z_\epsilon} e^{-H/\epsilon} \left\{ \frac{\epsilon^{C_2 K^2}}{\eta^d} e^{C_3 \eta/\epsilon} + o_\epsilon(1) \epsilon^{d/2} \left[ \left( \frac{\eta}{\epsilon} \right)^2 + \frac{\eta}{\epsilon} \right] \left( 1 + e^{C_4 \eta \delta/\epsilon} \right) \right\}. \end{aligned}$$

A similar estimate holds for  $\Phi_{p_\epsilon^{(\eta)}}, \Phi_{(p_\epsilon^*)^{(\eta)}}$  and  $\Phi_{(p_\epsilon^*)^{(\eta)}}$ .

Since  $p_\epsilon$  is discontinuous along  $\partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon$ , the function  $p_\epsilon^{(\eta)}$  has a bump around this boundary. The first term in the bracket takes this into account. Taking  $\eta = \epsilon$  and  $K$  a large enough real number, it follows from Lemma 5.4 that

$$\|\Phi_{p_\epsilon^{(\epsilon)}} - \Theta_{\mathbf{q}_\epsilon}\|^2 = o_\epsilon(1) T_\epsilon. \quad (5.11)$$

**Lemma 5.5.** *We have that*

$$\langle \Theta_{\mathbf{q}_\epsilon^*}, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle = -[1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0}). \quad (5.12)$$

The same estimate holds for  $\Theta_{\mathbf{q}_\epsilon^*}$ .



## 6. THE EQUILIBRIUM POTENTIAL

The main result of this section establishes a pointwise bound on the equilibrium potential between two open sets. We start recalling some classical estimates on the solutions of elliptic equations.

Fix  $0 < \alpha < 1$ . Unless otherwise stated, throughout this section  $\Omega \subset \mathbb{R}^d$  is a domain with boundary in  $C^{2,\alpha}$ ,  $\mathbf{g}$  a function in  $L^2(\Omega) \cap C^\alpha(\overline{\Omega})$  and  $\mathbf{b}$  a function in  $W^{1,2}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ , where the reference measure is  $Z_\epsilon^{-1} \exp\{-U(x)/\epsilon\} dx$ . We examine the Dirichlet problem (2.6) with  $\mathcal{L}$  replaced by  $\mathcal{L}_\epsilon$ .

**Harnack and Hölder estimates.** In this subsection,  $\Omega \subset \mathbb{R}^d$  represents a bounded domain and  $W^{2,p}(\Omega)$ ,  $p \geq 1$ , the space of twice weakly differentiable functions whose derivatives of order  $n \leq 2$  are in  $L^p(\Omega)$ .

Since  $\mathbb{M}$  is a positive-definite matrix, there exist  $0 < \lambda < \Lambda$  such that

$$\lambda \|\mathbf{x}\|^2 \leq \mathbf{x} \cdot \mathbb{M} \mathbf{x} \leq \Lambda \|\mathbf{x}\|^2 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d. \quad (6.1)$$

Clearly,  $\gamma = \Lambda/\lambda < \infty$ . For a domain  $\Omega \subset \mathbb{R}^d$ , let

$$\nu_\Omega = \frac{1}{\epsilon^2} \frac{\|\mathbb{M}\|_2^2}{\lambda^2} \sup_{\mathbf{x} \in \Omega} \|(\nabla U)(\mathbf{x})\|^2, \quad (6.2)$$

where  $\|(\nabla U)(\mathbf{x})\|^2 = \sum_j (\partial_{x_j} U)(\mathbf{x})^2$ ,  $\|\mathbb{M}\|_2^2 = \sum_{j,k} \mathbb{M}_{j,k}^2$ .

The Harnack inequality presented in the next result is [12, Corollary 9.25]. Denote by  $B_r(\mathbf{x})$  the open ball of radius  $r > 0$  centered at  $\mathbf{x} \in \mathbb{R}^d$ .

**Lemma 6.1.** *Let  $u \in W^{2,d}(\Omega)$  be a non-negative function which satisfies the equation  $\mathcal{L}_\epsilon u = 0$  in  $\Omega$ . Suppose that  $B_{2R}(\mathbf{x}) \subset \Omega$  for some  $R > 0$ ,  $\mathbf{x} \in \Omega$ . Then, there exists a constant  $C_0 = C_0(d, \gamma, \nu_\Omega R^2) < \infty$  such that*

$$\sup_{\mathbf{x} \in B_R(\mathbf{x})} u(\mathbf{x}) \leq C_0 \inf_{\mathbf{x} \in B_R(\mathbf{x})} u(\mathbf{x}).$$

Denote by  $\text{osc}(u, \mathcal{A})$  the oscillation of a function  $u : \mathcal{A} \rightarrow \mathbb{R}$  in the set  $\mathcal{A}$ :  $\text{osc}(u, \mathcal{A}) = \sup_{\mathbf{x} \in \mathcal{A}} u(\mathbf{x}) - \inf_{\mathbf{x} \in \mathcal{A}} u(\mathbf{x})$ . The Hölder estimate stated below is [12, Corollary 9.24].

**Lemma 6.2.** *Let  $u \in W^{2,d}(\Omega)$  satisfy the equation  $\mathcal{L}_\epsilon u = f$  in  $\Omega$  for some  $f \in L^d(\Omega)$ . Suppose that  $B_{R_0}(\mathbf{x}) \subset \Omega$  for some  $R_0 > 0$ ,  $\mathbf{x} \in \Omega$ . Then, there exist constants  $C_0 = C_0(d, \gamma, \nu_\Omega R_0^2) < \infty$ ,  $\alpha = \alpha(d, \gamma, \nu_\Omega R_0^2) > 0$  such that for all  $R \leq R_0$ ,*

$$\text{osc}(u, B_R(\mathbf{x})) \leq C_0 \left( \frac{R}{R_0} \right)^\alpha \left( \text{osc}(u, B_{R_0}(\mathbf{x})) + R_0 \|f\|_{d, B_{R_0}(\mathbf{x})} \right),$$

where  $\|f\|_{d, B_{R_0}(\mathbf{x})}$  stands for the  $L^d(B_{R_0}(\mathbf{x}))$  norm of  $f$ .

**The Green function.** We present in this subsection several properties of the Green function associated to the boundary-value problem (2.6). We do not assume  $\Omega \subset \mathbb{R}^d$  to be bounded.

By the assumptions (P4) and by Theorems 6.1.3, 4.2.1 (ii), 4.2.5 in [21], the generator  $\mathcal{L}_\epsilon$  possesses a non-negative Green function, denoted by  $G_\Omega : \Omega \times \Omega \rightarrow \mathbb{R}_+$ , such that for each  $\mathbf{y} \in \Omega$ ,

$$G_\Omega(\cdot, \mathbf{y}) \in C^{2,\alpha}(\Omega \setminus \{\mathbf{y}\}), \quad \mathcal{L}_\epsilon G_\Omega(\cdot, \mathbf{y}) = 0 \text{ on } \Omega \setminus \{\mathbf{y}\}. \quad (6.3)$$

The solutions of the boundary-value problem (2.6) can be represented in terms of the Green function. Next result follows from hypothesis (P4), which guarantees

that the process is positive recurrent, and Theorems 3.6.4 and 4.3.7 in [21] with  $\lambda = 0$ .

**Lemma 6.3.** *Assume that  $\Omega$  has a  $C^{2,\alpha}$ -boundary for some  $0 < \alpha < 1$ . Then, for any function  $g$  in  $C^\alpha(\bar{\Omega}) \cap L^2(\Omega)$  which vanishes at  $\partial\Omega$ , the function*

$$f(\mathbf{x}) = \int_{\Omega} G_{\Omega}(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y}$$

*belongs to  $C^{2,\alpha}(\bar{\Omega})$  and is the unique solution of the problem (2.6) with  $\mathbf{g} = g$ ,  $\mathbf{b} = 0$ .*

The previous result asserts that the Green function, as an operator, is the inverse of  $-\mathcal{L}_{\epsilon}$ . In particular, it inherits the dual properties of the generator. More precisely, if we denote by  $G_{\Omega}^*$  the Green function of the adjoint generator  $\mathcal{L}_{\epsilon}^*$ , it follows from the previous lemma that  $G_{\Omega}^*$  is the adjoint of  $G_{\Omega}$  in  $L^2(\mu_{\epsilon})$  so that

$$e^{-U(\mathbf{x})/\epsilon} G_{\Omega}(\mathbf{x}, \mathbf{y}) = e^{-U(\mathbf{y})/\epsilon} G_{\Omega}^*(\mathbf{y}, \mathbf{x}), \quad \mathbf{x} \neq \mathbf{y} \in \bar{\Omega}. \quad (6.4)$$

By Lemma 6.3,

$$G_{\Omega}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for all } \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega. \quad (6.5)$$

On the other hand, by (6.4) and (6.5) for  $G_{\Omega}^*$  in place of  $G_{\Omega}$ ,

$$G_{\Omega}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for all } \mathbf{x} \in \Omega, \mathbf{y} \in \partial\Omega. \quad (6.6)$$

Of course, all previous properties are in force for the adjoint Green function  $G_{\Omega}^*$ .

Next result is Theorem 4.2.8 in [21].

**Lemma 6.4.** *For each compact set  $\mathcal{K} \subset \Omega$ , there exist constants  $0 < c_1 < c_2 < \infty$  and  $r_0 \in (0, 1)$  such that for each  $\mathbf{x} \in \mathcal{K}$ ,*

$$c_1 \|\mathbf{y} - \mathbf{x}\|^{2-d} \leq G_{\Omega}(\mathbf{x}, \mathbf{y}) \leq c_2 \|\mathbf{y} - \mathbf{x}\|^{2-d}$$

*for all  $\|\mathbf{y} - \mathbf{x}\| < r_0$  if  $d \geq 3$ , and*

$$-c_1 \log \|\mathbf{y} - \mathbf{x}\| \leq G_{\Omega}(\mathbf{x}, \mathbf{y}) \leq -c_2 \log \|\mathbf{y} - \mathbf{x}\|$$

*for all  $\|\mathbf{y} - \mathbf{x}\| < r_0$  if  $d = 2$ .*

By (6.3), the function  $G_{\Omega}(\cdot, \mathbf{x})$  is harmonic on  $\Omega \setminus \{\mathbf{x}\}$ , and, by Lemma 6.4, it diverges at  $\mathbf{x}$ . The next lemma turns rigorous the formal identity  $[\mathcal{L}_{\epsilon} G_{\Omega}(\cdot, \mathbf{x})](\mathbf{y}) = -\delta_{\mathbf{x}}(\mathbf{y})$ , where  $\delta_{\mathbf{x}}$  is the Dirac delta function at  $\mathbf{x}$ .

**Lemma 6.5.** *Assume that  $\Omega$  has a  $C^{2,\alpha}$ -boundary for some  $0 < \alpha < 1$ , and let  $f$  be a function in  $C^{2,\alpha}(\bar{\Omega})$ . Then, for all  $\mathbf{x} \in \Omega$ ,*

$$f(\mathbf{x}) = \lim_{\delta \rightarrow 0} \epsilon \int_{\partial B_{\delta}(\mathbf{x})} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} f(\mathbf{y}) \mathbb{M}^{\dagger}(\nabla G_{\Omega}^*)(\mathbf{y}) \cdot \mathbf{n}_{B_{\delta}(\mathbf{x})^c}(\mathbf{y}) \sigma(d\mathbf{y}).$$

*Proof.* Fix  $\mathbf{x} \in \Omega$  and modify  $f$  outside a neighborhood of  $\mathbf{x}$  for  $f$  to vanish at  $\partial\Omega$ . We first claim that

$$f(\mathbf{x}) = - \int_{\Omega} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} G_{\Omega}^*(\mathbf{y}, \mathbf{x}) (\mathcal{L}_{\epsilon} f)(\mathbf{y}) d\mathbf{y}.$$

To prove this identity, denote by  $h$  the function defined by integral on the right hand side. Applying (6.4), we may replace  $\exp\{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon\} G_{\Omega}^*(\mathbf{y}, \mathbf{x})$  by  $G_{\Omega}(\mathbf{x}, \mathbf{y})$ . By assumption,  $\mathcal{L}_{\epsilon} f$  belongs to  $C^\alpha(\bar{\Omega})$  and vanishes at  $\partial\Omega$ . Therefore, by Lemma 6.3,  $h$  is the unique solution of (2.6) with  $\mathbf{g} = \mathcal{L}_{\epsilon} f$  and  $\mathbf{b} = 0$ . Since  $-f$  solves the same equation, by uniqueness,  $h = -f$ , proving the identity.

By Lemma 6.4, the integral on the right-hand side of the previous displayed equation is equal to

$$\lim_{\delta \rightarrow 0} \int_{\Omega \setminus B_\delta(\mathbf{x})} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} G_\Omega^*(\mathbf{y}, \mathbf{x}) (\mathcal{L}_\epsilon f)(\mathbf{y}) d\mathbf{y}.$$

By the divergence theorem and since  $G_\Omega^*(\cdot, \mathbf{x})$  vanishes at  $\partial\Omega$ , the previous integral is equal to

$$\begin{aligned} & \epsilon \int_{\partial B_\delta(\mathbf{x})} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} G_\Omega^*(\mathbf{y}, \mathbf{x}) \mathbb{M} \nabla f(\mathbf{y}) \cdot \mathbf{n}_{B_\delta(\mathbf{x})^c}(\mathbf{y}) \sigma(d\mathbf{y}) \\ & - \epsilon \int_{\Omega \setminus B_\delta(\mathbf{x})} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} \mathbb{M}^\dagger \nabla G_\Omega^*(\mathbf{y}, \mathbf{x}) \nabla f(\mathbf{y}) d\mathbf{y}. \end{aligned} \quad (6.7)$$

By Lemma 6.4, the first integral vanishes as  $\delta \rightarrow 0$ . By the divergence theorem, since  $f$  vanishes on  $\partial\Omega$  and since  $[\mathcal{L}_\epsilon^* G_\Omega^*(\cdot, \mathbf{x})](\mathbf{y}) = 0$  on  $\Omega \setminus \{\mathbf{x}\}$ , the second one is equal to

$$- \epsilon \int_{\partial B_\delta(\mathbf{x})} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} f(\mathbf{y}) \mathbb{M}^\dagger \nabla G_\Omega^*(\mathbf{y}, \mathbf{x}) \cdot \mathbf{n}_{B_\delta(\mathbf{x})^c}(\mathbf{y}) \sigma(d\mathbf{y}).$$

This completes the proof of the lemma.  $\square$

**Lemma 6.6.** *Assume that  $\Omega$  has a  $C^{2,\alpha}$ -boundary for some  $0 < \alpha < 1$ , and let  $b$  be a function in  $C^{2,\alpha}(\overline{\Omega}) \cap W^{1,2}(\Omega)$ . The unique solution in  $C^{2,\alpha}(\overline{\Omega})$  of the Dirichlet problem (2.6) with  $\mathbf{g} = 0$ ,  $\mathbf{b} = b$ , denoted by  $f$ , can be represented as*

$$f(\mathbf{x}) = - \epsilon \int_{\partial\Omega} b(\mathbf{y}) e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} \mathbb{M}^\dagger \nabla G_\Omega^*(\mathbf{y}, \mathbf{x}) \cdot \mathbf{n}_\Omega(\mathbf{y}) \sigma(d\mathbf{y}), \quad \mathbf{x} \in \Omega.$$

*Proof.* Fix  $\mathbf{x}$  in  $\Omega$ , and let  $h(\mathbf{y}) = G_\Omega^*(\mathbf{y}, \mathbf{x})$ . By (6.3) and (6.5),  $h \in C^{2,\alpha}(\Omega \setminus \{\mathbf{x}\})$ ,  $(\mathcal{L}_\epsilon^* h)(\mathbf{y}) = 0$ ,  $\mathbf{y} \in \Omega \setminus \{\mathbf{x}\}$ , and  $h(\mathbf{y}) = 0$ ,  $\mathbf{y} \in \partial\Omega$ .

Denote by  $f$  the solution of the Dirichlet problem (2.6) with  $\mathbf{g} = 0$ ,  $\mathbf{b} = b$ . Fix  $\delta > 0$ . Since  $\mathcal{L}_\epsilon f = 0$  and since  $h$  vanishes on  $\partial\Omega$ , by the divergence theorem,

$$\begin{aligned} 0 &= \int_{\Omega \setminus B_\delta(\mathbf{x})} h(\mathbf{y}) \nabla \cdot (e^{-U/\epsilon} \mathbb{M} \nabla f)(\mathbf{y}) d\mathbf{y} \\ &= - \int_{\Omega \setminus B_\delta(\mathbf{x})} e^{-U(\mathbf{y})/\epsilon} \mathbb{M}^\dagger \nabla h(\mathbf{y}) \cdot \nabla f(\mathbf{y}) d\mathbf{y} + o_\delta(1). \end{aligned}$$

The expression  $o_\delta(1)$  comes from the integral on  $\partial B_\delta(\mathbf{x})$ , which vanishes as  $\delta \rightarrow 0$  as we have seen in (6.7).

Applying the divergence theorem once more, since  $\mathcal{L}_\epsilon^* h = 0$  on  $\Omega \setminus B_\delta(\mathbf{x})$ , the right hand side of the previous identity is equal to

$$\begin{aligned} & - \int_{\partial B_\delta(\mathbf{x})} f(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} \mathbb{M}^\dagger \nabla h(\mathbf{y}) \cdot \mathbf{n}_{B_\delta(\mathbf{x})^c}(\mathbf{y}) \sigma(d\mathbf{y}) \\ & - \int_{\partial\Omega} f(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} \mathbb{M}^\dagger \nabla h(\mathbf{y}) \cdot \mathbf{n}_\Omega(\mathbf{y}) \sigma(d\mathbf{y}) + o_\delta(1). \end{aligned}$$

As  $f$  belongs to  $C^{2,\alpha}(\overline{\Omega})$  and is equal to  $b$  on  $\partial\Omega$ , by Lemma 6.5, letting  $\delta \rightarrow 0$ , this sum converges to

$$- \epsilon^{-1} f(\mathbf{x}) e^{-U(\mathbf{x})/\epsilon} - \int_{\partial\Omega} b(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} \mathbb{M}^\dagger \nabla h(\mathbf{y}) \cdot \mathbf{n}_\Omega(\mathbf{y}) \sigma(d\mathbf{y}).$$

This completes the proof of the lemma.  $\square$

**The equilibrium potential.** In this subsection, we establish a bound on the harmonic function in terms of capacities and simple bounds for the capacity between two sets. Together these estimate provide a useful bound on the harmonic function.

Let  $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^d$  be two open subsets of  $\mathbb{R}^d$  satisfying the assumptions S, and let  $\Omega = (\overline{\mathcal{A}} \cup \overline{\mathcal{B}})^c$ . The next result presents a formula for the equilibrium potential. The same proof provides an identity for  $h_{\mathcal{A}, \mathcal{B}}^*$  in place of  $h_{\mathcal{A}, \mathcal{B}}$ .

**Lemma 6.7.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be open sets satisfying assumptions S. Then, for all  $\mathbf{x} \notin \mathcal{B}$ ,*

$$h_{\mathcal{A}, \mathcal{B}}(\mathbf{x}) = \epsilon \int_{\partial \mathcal{A}} G_{\mathcal{B}^c}(\mathbf{x}, \mathbf{y}) \mathbb{M} \nabla h_{\mathcal{A}, \mathcal{B}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{A}^c}(\mathbf{y}) \sigma(d\mathbf{y}).$$

*Proof.* Consider the integral on the right-hand side. Since  $G_{\mathcal{B}^c}(\mathbf{x}, \cdot)$  vanishes at  $\partial \mathcal{B}$ , we may extend the integral to  $\partial \mathcal{A} \cup \partial \mathcal{B}$ . By (6.4), we may replace  $G_{\mathcal{B}^c}(\mathbf{x}, \mathbf{y})$  by  $\exp\{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon\} G_{\mathcal{B}^c}^*(\mathbf{y}, \mathbf{x})$ . On the other hand, as  $h_{\mathcal{A}, \mathcal{B}} = 1 - h_{\mathcal{B}, \mathcal{A}}$ , we may also replace  $\nabla h_{\mathcal{A}, \mathcal{B}}$  by  $-\nabla h_{\mathcal{B}, \mathcal{A}}$ . After these modifications, the integral appearing in the statement of the lemma becomes

$$- \epsilon \int_{\partial \mathcal{A} \cup \partial \mathcal{B}} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} G_{\mathcal{B}^c}^*(\mathbf{y}, \mathbf{x}) \mathbb{M} \nabla h_{\mathcal{B}, \mathcal{A}}(\mathbf{y}) \cdot \mathbf{n}_{(\mathcal{A} \cup \mathcal{B})^c}(\mathbf{y}) \sigma(d\mathbf{y}).$$

In the argument below, as we did in the two previous lemmata, we need to remove from the integration region a ball  $B_\delta(\mathbf{x})$  and let  $\delta \rightarrow 0$ . As the argument should be clear at this point, we ignore the singularity of the Green function at  $\mathbf{x}$ . By the divergence theorem, and since  $\mathcal{L}_\epsilon h_{\mathcal{B}, \mathcal{A}} = 0$  on  $(\mathcal{A} \cup \mathcal{B})^c$ , this expression is equal to

$$- \epsilon \int_{(\mathcal{A} \cup \mathcal{B})^c} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} \mathbb{M}^\dagger \nabla G_{\mathcal{B}^c}^*(\mathbf{y}, \mathbf{x}) \nabla h_{\mathcal{B}, \mathcal{A}}(\mathbf{y}) d\mathbf{y}.$$

Applying the divergence theorem a second time, as  $\mathcal{L}_\epsilon^* G_{\mathcal{B}^c}^*(\cdot, \mathbf{x}) = -\delta_{\mathbf{x}}(\cdot)$  and  $h_{\mathcal{B}, \mathcal{A}} = \mathbf{1}\{\mathcal{B}\}$  on  $\partial \mathcal{A} \cup \partial \mathcal{B}$ , this expression becomes

$$- h_{\mathcal{B}, \mathcal{A}}(\mathbf{x}) - \epsilon \int_{\partial \mathcal{B}} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} \mathbb{M}^\dagger \nabla G_{\mathcal{B}^c}^*(\mathbf{y}, \mathbf{x}) \cdot \mathbf{n}_{\mathcal{B}^c}(\mathbf{y}) \sigma(d\mathbf{y}).$$

By Lemma 6.6 the integral is equal to  $f(\mathbf{x})$  where  $f$  is the solution of (2.6) with  $\Omega = \mathcal{B}^c$ ,  $\mathbf{g} = 0$  and  $\mathbf{b} = 1$ . Since the solution of this equation is equal to 1, the previous expression is equal to  $1 - h_{\mathcal{B}, \mathcal{A}}(\mathbf{x}) = h_{\mathcal{A}, \mathcal{B}}(\mathbf{x})$ , as claimed.  $\square$

In the present context, the harmonic measure  $\nu_{\mathcal{A}, \mathcal{B}}$ , introduced in (3.12), is the probability measure on  $\partial \mathcal{A}$  given by

$$\nu_{\mathcal{A}, \mathcal{B}}(d\mathbf{y}) = \frac{\epsilon}{Z_\epsilon \text{cap}(\mathcal{A}, \mathcal{B})} e^{-U(\mathbf{y})/\epsilon} \mathbb{M}^\dagger \nabla h_{\mathcal{A}, \mathcal{B}}^*(\mathbf{y}) \cdot \mathbf{n}_\Omega(\mathbf{y}) \sigma(d\mathbf{y}). \quad (6.8)$$

In particular, in view of (6.4), in terms of the harmonic measure, the formula for  $h_{\mathcal{A}, \mathcal{B}}^*$  becomes

$$h_{\mathcal{A}, \mathcal{B}}^*(\mathbf{x}) = Z_\epsilon \text{cap}(\mathcal{A}, \mathcal{B}) e^{U(\mathbf{x})/\epsilon} \int_{\partial \mathcal{A}} G_{\mathcal{B}^c}(\mathbf{y}, \mathbf{x}) \nu_{\mathcal{A}, \mathcal{B}}(d\mathbf{y}), \quad \mathbf{x} \notin \mathcal{B}.$$

Therefore, since  $h_{\mathcal{A}, \mathcal{B}}^*(\mathbf{x}) = 1$  for  $\mathbf{x} \in \mathcal{A}$  and since the harmonic measure is a probability measure, we obtain that

$$\inf_{\mathbf{y} \in \partial \mathcal{A}} G_{\mathcal{B}^c}(\mathbf{y}, \mathbf{x}) \leq \frac{e^{-U(\mathbf{x})/\epsilon}}{Z_\epsilon \text{cap}(\mathcal{A}, \mathcal{B})}, \quad \mathbf{x} \in \mathcal{A}. \quad (6.9)$$

**Lemma 6.8.** *Let  $\mathcal{D}$  be an open set with a  $C^{2,\alpha}$ -boundary for some  $0 < \alpha < 1$ . Fix  $\mathbf{x} \notin \overline{\mathcal{D}}$ . For every  $r > 0$  there exists a finite constant  $C_0$ , depending only on  $r$  and  $U$ , such that for all  $0 < \epsilon < \epsilon_0 = d(\mathbf{x}, \mathcal{D})/2r$ ,*

$$\sup_{\mathbf{y} \in \partial B_{r\epsilon}(\mathbf{x})} G_{\mathcal{D}^c}(\mathbf{y}, \mathbf{x}) \leq C_0 \inf_{\mathbf{y} \in \partial B_{r\epsilon}(\mathbf{x})} G_{\mathcal{D}^c}(\mathbf{y}, \mathbf{x}).$$

*Proof.* The proof is a well-known application of the Harnack inequality, see e.g. [8, Lemma 4.6]. Note that the supremum and infimum are carried over the boundary of  $B_{r\epsilon}(\mathbf{x})$ . The result does not hold if this boundary is replaced by the ball since the Green function diverges on the diagonal, as stated in Lemma 6.4.  $\square$

**Proposition 6.9.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be open sets satisfying assumptions S. Fix  $\mathbf{x} \notin \overline{\mathcal{A} \cup \mathcal{B}}$  and  $r > 0$ . Let  $\epsilon_0 = d(\mathbf{x}, \mathcal{A} \cup \mathcal{B})/2r$ . There exists a finite constant  $C_0$ , depending only on  $r$  and  $U$ , such that for all  $\epsilon < \epsilon_0$ ,*

$$h_{\mathcal{A}, \mathcal{B}}(\mathbf{x}) \leq C_0 \frac{\text{cap}(B_{r\epsilon}(\mathbf{x}), \mathcal{A})}{\text{cap}(B_{r\epsilon}(\mathbf{x}), \mathcal{A} \cup \mathcal{B})}.$$

*Proof.* Fix two open sets  $\mathcal{A}, \mathcal{B}$  satisfying assumptions S and  $\mathbf{x} \notin \overline{\mathcal{A} \cup \mathcal{B}}$ . By Lemma 6.6,

$$h_{\mathcal{A}, \mathcal{B}}(\mathbf{x}) = -\epsilon \int_{\partial \mathcal{A}} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} \mathbb{M}^\dagger \nabla G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x}) \cdot \mathbf{n}_{\mathcal{A}^c}(\mathbf{y}) \sigma(d\mathbf{y}),$$

Let  $\mathcal{C}$  be an open set with a smooth boundary and such that  $d(\mathcal{C}, \mathcal{A} \cup \mathcal{B}) > 0$ ,  $\mathbf{x} \in \mathcal{C}$ . Since  $h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}} = 1$  on  $\partial \mathcal{A}$  we may add  $h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}$  inside the integral and then extend the integral to  $\partial \mathcal{A} \cup \partial \mathcal{B} \cup \partial \mathcal{C}$ . By the divergence theorem, since  $\mathbf{x} \in \mathcal{C}$  and  $\mathcal{L}_\epsilon^* G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\cdot, \mathbf{x}) = 0$  on  $(\mathcal{A} \cup \mathcal{B} \cup \overline{\mathcal{C}})^c$ , the previous expression is equal to

$$-\epsilon \int_{(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})^c} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} \nabla h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}(\mathbf{y}) \cdot \mathbb{M}^\dagger \nabla G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x}) d\mathbf{y}.$$

Applying once more the divergence theorem, and since  $\mathcal{L}_\epsilon h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}} = 0$  on  $(\mathcal{A} \cup \mathcal{B} \cup \overline{\mathcal{C}})^c$ , the previous expression is equal to

$$-\epsilon \int_{\partial \mathcal{C}} e^{[U(\mathbf{x}) - U(\mathbf{y})]/\epsilon} G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x}) \mathbb{M} \nabla h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{C}^c}(\mathbf{y}) \sigma(d\mathbf{y}).$$

Note that the integration is carried over  $\partial \mathcal{C}$  because  $G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x})$  vanishes on  $\partial \mathcal{A} \cup \partial \mathcal{B}$ .

We prove below that on  $\partial \mathcal{C}$ ,  $\mathbb{M} \nabla h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{C}^c}(\mathbf{y}) \geq \mathbb{M} \nabla h_{\mathcal{A}, \mathcal{C}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{C}^c}(\mathbf{y})$ . After replacing  $\nabla h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}$  by  $\nabla h_{\mathcal{A}, \mathcal{C}}$  in the previous displayed equation and observing that  $\nabla h_{\mathcal{A}, \mathcal{C}} = -\nabla h_{\mathcal{C}, \mathcal{A}}$ , we conclude that

$$\begin{aligned} h_{\mathcal{A}, \mathcal{B}}(\mathbf{x}) &\leq \epsilon e^{U(\mathbf{x})/\epsilon} \int_{\partial \mathcal{C}} e^{-U(\mathbf{y})/\epsilon} G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x}) \mathbb{M} \nabla h_{\mathcal{C}, \mathcal{A}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{C}^c}(\mathbf{y}) \sigma(d\mathbf{y}) \\ &\leq \epsilon e^{U(\mathbf{x})/\epsilon} \sup_{\mathbf{y} \in \partial \mathcal{C}} G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x}) \int_{\partial \mathcal{C}} e^{-U(\mathbf{y})/\epsilon} \mathbb{M} \nabla h_{\mathcal{C}, \mathcal{A}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{C}^c}(\mathbf{y}) \sigma(d\mathbf{y}) \\ &= e^{U(\mathbf{x})/\epsilon} \sup_{\mathbf{y} \in \partial \mathcal{C}} G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x}) Z_\epsilon \text{cap}(\mathcal{C}, \mathcal{A}). \end{aligned}$$

In the last step we used the formula (2.8) for the capacity.

Fix  $r > 0$ , let  $\epsilon_0 = d(\mathbf{x}, \mathcal{A} \cup \mathcal{B})/2r$ , and set  $\mathcal{C} = B_{r\epsilon}(\mathbf{x})$ . By Lemma 6.8 with  $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$ , there exists a finite constant  $C_0 = C_0(r)$  such that

$$h_{\mathcal{A}, \mathcal{B}}(\mathbf{x}) \leq C_0 e^{U(\mathbf{x})/\epsilon} \inf_{\mathbf{y} \in \partial B_{r\epsilon}(\mathbf{x})} G_{(\mathcal{A} \cup \mathcal{B})^c}^*(\mathbf{y}, \mathbf{x}) Z_\epsilon \text{cap}(B_{r\epsilon}(\mathbf{x}), \mathcal{A}).$$

To complete the proof of the lemma, it remains to recall estimate (6.9).

It remains to show that  $\mathbb{M} \nabla h_{\mathcal{A}, \mathcal{C}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{C}^c}(\mathbf{y}) \leq \mathbb{M} \nabla h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}(\mathbf{y}) \cdot \mathbf{n}_{\mathcal{C}^c}(\mathbf{y})$  for  $\mathbf{y} \in \partial \mathcal{C}$ . Fix  $\mathbf{x} \in \partial \mathcal{C}$ . The vector  $\mathbb{M}^\dagger \mathbf{n}_{\mathcal{C}^c}(\mathbf{x})$  points inward to  $\mathcal{C}$  because  $\mathbf{n}_{\mathcal{C}^c}(\mathbf{x}) \cdot \mathbb{M}^\dagger \mathbf{n}_{\mathcal{C}^c}(\mathbf{x}) = \mathbf{n}_{\mathcal{C}^c}(\mathbf{x}) \cdot \mathbb{S} \mathbf{n}_{\mathcal{C}^c}(\mathbf{x}) > 0$ . In particular, for  $\delta$  small enough,  $\mathbf{x} - \delta \mathbb{M}^\dagger \mathbf{n}_{\mathcal{C}^c}(\mathbf{x}) \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})^c$ . Since, by (3.2),  $h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}} \leq h_{\mathcal{A}, \mathcal{C}}$  on this set,

$$h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}(\mathbf{x} - \delta \mathbb{M}^\dagger \mathbf{n}_{\mathcal{C}^c}(\mathbf{x})) \leq h_{\mathcal{A}, \mathcal{C}}(\mathbf{x} - \delta \mathbb{M}^\dagger \mathbf{n}_{\mathcal{C}^c}(\mathbf{x}))$$

for  $\delta$  small enough. Subtracting  $h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}(\mathbf{x}) = h_{\mathcal{A}, \mathcal{C}}(\mathbf{x}) = 0$  on both sides, dividing by  $\delta$  and letting  $\delta \rightarrow 0$  yields that

$$\mathbf{n}_{\mathcal{C}^c}(\mathbf{x}) \cdot \mathbb{M} \nabla h_{\mathcal{A}, \mathcal{B} \cup \mathcal{C}}(\mathbf{x}) \geq \mathbf{n}_{\mathcal{C}^c}(\mathbf{x}) \cdot \mathbb{M} \nabla h_{\mathcal{A}, \mathcal{C}}(\mathbf{x}),$$

as claimed.  $\square$

**Lemma 6.10.** *There exists a finite constant  $C_0$  and  $\epsilon_0 > 0$  such that for all  $\mathbf{y} \in \mathcal{W}_2$ ,  $0 < \epsilon < \epsilon_0$ ,*

$$\text{cap}(B_\epsilon(\mathbf{y}), \mathcal{V}_1) \leq \frac{C_0}{Z_\epsilon} e^{-H/\epsilon}, \quad \text{cap}(B_\epsilon(\mathbf{y}), \mathcal{V}_2) \geq C_0 \frac{\epsilon^d}{Z_\epsilon} e^{-U(\mathbf{y})/\epsilon}.$$

*Proof.* The generator  $\mathcal{L}_\epsilon$  satisfies a sector condition with constant  $\Lambda/\lambda$ . Indeed, by Schwarz inequality and by (6.1), for any smooth functions  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\langle f, \mathcal{L}_\epsilon g \rangle_{\mu_\epsilon}^2 \leq \frac{\Lambda}{\lambda} \langle f, (-\mathcal{L}_\epsilon) f \rangle_{\mu_\epsilon}^2 \langle g, (-\mathcal{L}_\epsilon) g \rangle_{\mu_\epsilon}^2.$$

It follows from Lemmata 2.5 and 2.6 in [13] that the capacity between two sets can be estimated from below and from above by the capacity associated to the symmetric operator  $(1/2)(\mathcal{L}_\epsilon + \mathcal{L}_\epsilon^*)$ . Denote by  $\text{cap}^s(\mathcal{A}, \mathcal{B})$  the capacity between the sets  $\mathcal{A}, \mathcal{B}$  for the symmetric process.

We start with the upper bound. Let  $\overline{\mathcal{W}}_2^t = \{\mathbf{x} : d(\mathbf{x}, \mathcal{W}_2) \leq t\}$ ,  $t \geq 0$ . Let  $\epsilon_0 > 0$  such that  $\mathcal{V}_1 \cap \overline{\mathcal{W}}_2^{2\epsilon_0} = \emptyset$ , and fix  $0 < \epsilon < \epsilon_0$ . There exist a smooth function  $h_\epsilon$  and a finite constant  $C_0$ , independent on  $\epsilon$ , such that  $h_\epsilon \equiv 1$  on  $\overline{\mathcal{W}}_2^\epsilon$ ,  $h_\epsilon \equiv 0$  on  $(\overline{\mathcal{W}}_2^{2\epsilon})^c$ , and

$$\|\nabla h_\epsilon(\mathbf{x})\| \leq C_0 \epsilon^{-1} \quad \text{for all } \mathbf{x} \in \overline{\mathcal{W}}_2^{2\epsilon} \setminus \overline{\mathcal{W}}_2^\epsilon.$$

Then, since  $B_\epsilon(\mathbf{y}) \subset \overline{\mathcal{W}}_2^\epsilon$  and  $\mathcal{V}_1 \subset (\overline{\mathcal{W}}_2^{2\epsilon})^c$ , and since  $U(\mathbf{x}) = H + O(\epsilon)$  for all  $\mathbf{x} \in \mathcal{W}_2^{2\epsilon} \setminus \mathcal{W}_2^\epsilon$ , by the Dirichlet principle for reversible processes,

$$\begin{aligned} \text{cap}^s(B_\epsilon(\mathbf{y}), \mathcal{V}_1) &\leq \frac{\epsilon}{Z_\epsilon} \int_{\mathbb{R}^d} e^{-U(\mathbf{x})/\epsilon} \nabla h_\epsilon(\mathbf{x}) \cdot \mathbb{S} \nabla h_\epsilon(\mathbf{x}) d\mathbf{x} \\ &\leq C_0 \frac{\epsilon}{Z_\epsilon} e^{-H/\epsilon} \epsilon^{-2} \text{vol}(\overline{\mathcal{W}}_2^{2\epsilon} \setminus \overline{\mathcal{W}}_2^\epsilon) \leq \frac{C_0}{Z_\epsilon} e^{-H/\epsilon}. \end{aligned}$$

We turn to the lower bound, where we follow the argument of [8, Proposition 4.7]. Fix  $0 < \epsilon < 1$ . Let  $\omega(t)$  be a smooth path connecting  $\mathbf{y}$  to  $\mathbf{m}_2$  such that  $U(\omega(t))$  is decreasing in  $t$ , and  $\|\dot{\omega}(t)\| = 1$  for all  $t$ . Let  $D_\epsilon$  be a  $(d-1)$ -dimensional disk of radius  $\epsilon$  centered at origin. By the proof [8, Proposition 4.7] up to equation (4.26), we obtain that

$$\text{cap}^s(B_\epsilon(\mathbf{y}), \mathcal{V}_2) \geq \frac{\epsilon}{Z_\epsilon} \int_{D_\epsilon} d\mathbf{z}_\perp \left[ \int_0^{|\omega|} dt e^{U(\omega(t) + \mathbf{z}_\perp)/\epsilon} \right]^{-1}.$$

Let  $L_0 = \sup_{\mathbf{x} \in \overline{\mathcal{W}_2^1}} \|\nabla U(\mathbf{x})\|$ . As  $U(\omega(t))$  decreases in  $t$ ,

$$\int_0^{|\omega|} e^{U(\omega(t)+\mathbf{z}_\perp)/\epsilon} dt \leq e^{L_0} \int_0^{|\omega|} e^{U(\omega(t))/\epsilon} dt \leq e^{L_0} |\omega| e^{U(\mathbf{y})/\epsilon}.$$

Since the set  $\mathcal{W}_2$  is bounded, we can choose smooth paths with length  $|\omega|$  uniformly bounded. Hence, by the previous estimates,

$$\text{cap}^s(B_\epsilon(\mathbf{y}), \mathcal{V}_2) \geq C_0 \frac{\epsilon^d}{Z_\epsilon} e^{-U(\mathbf{y})/\epsilon},$$

as claimed.  $\square$

**Proposition 6.11.** *There exists a finite constant  $C_0$  and  $\epsilon_0 > 0$  such that for all  $\mathbf{y} \in \mathcal{W}_2$ ,  $0 < \epsilon < \epsilon_0$ ,*

$$h_{\mathcal{V}_1, \mathcal{V}_2}(\mathbf{y}) \leq C_0 \epsilon^{-d} e^{-[H-U(\mathbf{y})]/\epsilon}. \quad (6.10)$$

*Proof.* Fix  $\mathbf{y} \in \mathcal{W}_2$ . By Proposition 6.9 with  $r = 1$ , for all  $\epsilon$  small enough and since the capacity is monotone in its arguments,

$$h_{\mathcal{V}_1, \mathcal{V}_2}(\mathbf{y}) \leq C_0 \frac{\text{cap}(B_\epsilon(\mathbf{y}), \mathcal{V}_1)}{\text{cap}(B_\epsilon(\mathbf{y}), \mathcal{V}_2)}.$$

By Lemma 6.10, this expression is bounded above by the right hand side of (6.10) for all  $\epsilon$  small enough, as claimed.  $\square$

## 7. THE VECTOR FIELDS $\Theta_{\mathbf{q}_\epsilon}, \Theta_{\mathbf{q}_\epsilon}^*$

We prove in this section Lemmata 5.3, 5.4 and 5.5. Throughout this section,  $C_1, C_2, C_3$  represent large but finite positive constants, independent of the variables  $\epsilon$  and  $\eta$  introduced in Section 5, and whose value may change from line to line. Similarly,  $c_1, c_2$  represent small but positive constants with the same properties of  $C_1, C_2$ .

We start by recalling basic properties of the vector  $\mathbf{v}$  and the matrices  $\mathbb{M}, \mathbb{L}$ . Most of these results were proven in Section 4 of [17]. Recall that we write a vector  $\mathbf{u} \in \mathbb{R}^d$  as  $\sum_{1 \leq i \leq d} u_i \mathbf{e}_i$ , that we represent by  $\mathbf{v}$  the eigenvector of  $\mathbb{L}\mathbb{M}$  associated to the eigenvalue  $-\mu$ , and that we assumed  $v_1 > 0$ .

**Lemma 7.1.** *We have that*

$$\mathbf{v} \cdot \mathbb{L}^{-1} \mathbf{v} = -\frac{v_1^2}{\lambda_1} + \sum_{k=2}^d \frac{v_k^2}{\lambda_k} = -\frac{1}{\alpha}.$$

*Proof.* Since  $\mathbf{v}$  is the eigenvector of  $\mathbb{L}\mathbb{M}$  associated to the eigenvalue  $-\mu$ , by (5.2),

$$-\mathbf{v} \cdot \mathbb{L}^{-1} \mathbf{v} = -\mathbf{v} \cdot \mathbb{M}(\mathbb{L}\mathbb{M})^{-1} \mathbf{v} = \frac{1}{\mu} \mathbf{v} \cdot \mathbb{M} \mathbf{v} = \frac{1}{\alpha}.$$

$\square$

The next two results are Lemmata 4.1 and 4.2 of [17]. Denote by  $\mathbf{w}^\dagger$  the transpose of a vector  $\mathbf{w} \in \mathbb{R}^d$ .

**Lemma 7.2.** *The matrix  $\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger$  is positive definite and  $\det(\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger) = -\det \mathbb{L}$ .*

**Lemma 7.3.** *The matrix  $\mathbb{L} + \alpha \mathbf{v} \mathbf{v}^\dagger$  is non-negative definite and  $\det(\mathbb{L} + \alpha \mathbf{v} \mathbf{v}^\dagger) = 0$ . The null space of the matrix  $\mathbb{L} + \alpha \mathbf{v} \mathbf{v}^\dagger$  is one-dimensional and spanned by the vector  $\mathbb{L}^{-1} \mathbf{v}$ .*

**A. Proof of Lemma 5.3.** The proof of Lemma 5.3 is based on the following estimate.

**Lemma 7.4.** *We have that*

$$\int_{\mathcal{B}_\epsilon} \nabla p_\epsilon(\mathbf{z}) \cdot \mathbb{M} \nabla p_\epsilon(\mathbf{z}) e^{-(U(\mathbf{z})-H)/\epsilon} d\mathbf{z} = [1 + o_\epsilon(1)] (2\pi\epsilon)^{\frac{d}{2}-1} \omega(\mathbf{0}).$$

*Proof.* By the definition (5.3) of  $p_\epsilon$ ,

$$\nabla p_\epsilon(\mathbf{z}) = \frac{1}{C_\epsilon} \exp \left\{ -\frac{\alpha}{2\epsilon} (\mathbf{z} \cdot \mathbf{v})^2 \right\} \mathbf{v},$$

and by the Taylor expansion of the potential  $U$  around  $\mathbf{0}$ , on the set  $\mathcal{B}_\epsilon$ ,

$$U(\mathbf{z}) - H = (1/2) \mathbf{z} \cdot \mathbb{L} \mathbf{z} + O(\delta^3).$$

Since  $\exp\{\delta^3/\epsilon\} = 1 + o_\epsilon(1)$  and  $C_\epsilon = \sqrt{2\pi\epsilon/\alpha}$ , by (5.2) and by the two previous identities, the left hand side of the expression appearing in the statement of the lemma is equal to

$$\begin{aligned} & [1 + o_\epsilon(1)] \frac{\mathbf{v} \cdot \mathbb{M} \mathbf{v}}{C_\epsilon^2} \int_{\mathcal{B}_\epsilon} \exp \left\{ \frac{1}{2\epsilon} \mathbf{z} \cdot [\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z} \right\} d\mathbf{z} \\ &= [1 + o_\epsilon(1)] \frac{\mu}{2\pi\epsilon} \int_{\mathcal{B}_\epsilon} \exp \left\{ \frac{1}{2\epsilon} \mathbf{z} \cdot [\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z} \right\} d\mathbf{z}. \end{aligned}$$

It is easy to verify that

$$[-\delta, \delta] \times \prod_{i=2}^d \left[ -\sqrt{\frac{\lambda_1}{4(d-1)\lambda_i}}, \sqrt{\frac{\lambda_1}{4(d-1)\lambda_i}} \right] \subseteq \mathcal{B}_\epsilon.$$

Hence, by the change of coordinates  $\mathbf{y} = (1/\sqrt{\epsilon}) \mathbf{z}$ , and by Lemma 7.2, the last integral is equal to

$$[1 + o_\epsilon(1)] \frac{(2\pi\epsilon)^{d/2}}{\sqrt{-\det \mathbb{L}}}.$$

This completes the proof of the lemma since  $\omega(\mathbf{0}) = \mu/\sqrt{-\det \mathbb{L}}$ .  $\square$

We may now turn to the Proof of Lemma 5.3.

*Proof of Lemma 5.3.* By the definition of  $\Theta_{\mathbf{q}_\epsilon}$  and  $\Theta_{\mathbf{q}_\epsilon}^*$ , it is easy to check that

$$\left\| \frac{\Theta_{\mathbf{q}_\epsilon} + \Theta_{\mathbf{q}_\epsilon}^*}{2} \right\|^2 = \frac{\epsilon}{Z_\epsilon} \int_{\mathcal{B}_\epsilon} \nabla p_\epsilon(x) \cdot \mathbb{M} \nabla p_\epsilon(x) e^{-U(x)/\epsilon} dx.$$

At this point, the assertion of Lemma 5.3 follows from Lemma 7.4.  $\square$

**B. Proof of Lemma 5.4.** Define a mollified version of the vector field  $\mathbf{q}_\epsilon$  as  $\mathbf{q}_\epsilon^{(\eta)} = \mathbf{q}_\epsilon * \phi_\eta$ , where  $\eta = \eta(\epsilon)$  is such that  $\lim_{\epsilon \rightarrow 0} \eta(\epsilon)/\delta(\epsilon) = 0$ . Let  $\Theta_{\mathbf{q}_\epsilon^{(\eta)}}(\mathbf{z}) = \epsilon Z_\epsilon^{-1} e^{-U(\mathbf{z})/\epsilon} \mathbb{M}^\dagger \mathbf{q}_\epsilon^{(\eta)}(\mathbf{z})$ . By Young's inequality,

$$\|\Phi_{p_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon}\|^2 \leq 2 \|\Phi_{p_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon^{(\eta)}}\|^2 + 2 \|\Theta_{\mathbf{q}_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon}\|^2.$$

We estimate the two terms on the right hand side separately. Lemma 5.4 follows from Lemmata 7.5 and 7.6 below.



**Lemma 7.5.** *There exist finite constants  $C_1, c_2, C_3$ , such that*

$$\|\Phi_{p_\epsilon(\eta)} - \Theta_{q_\epsilon(\eta)}\|^2 \leq \frac{C_1}{Z_\epsilon} e^{-H/\epsilon} \frac{\epsilon^{c_2 K^2}}{\eta^d} e^{C_3 \eta/\epsilon}.$$

The proof of this lemma is divided in several steps. The crucial point is the control of the discontinuity of  $p_\epsilon$  along the boundary  $\partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon$ . For  $\mathbf{z} \in \partial\mathcal{X}_\epsilon$ , let  $\mathbf{n}(\mathbf{z})$  be the inner normal vector to  $\mathcal{X}_\epsilon$  at  $\mathbf{z}$  (and hence the outer normal vector to  $\mathcal{W}_1^\epsilon \cup \mathcal{W}_2^\epsilon \cup \mathcal{B}_\epsilon$ ). Similarly, for  $\mathbf{z} \in \partial\mathcal{B}_\epsilon \setminus \partial\mathcal{X}_\epsilon$ , let  $\mathbf{n}(\mathbf{z})$  be the outer normal vector to  $\mathcal{B}_\epsilon$  at  $\mathbf{z}$ . In this manner, the normal vector is defined for all  $\mathbf{z} \in \partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon$ .

Define the functions  $\mathfrak{d}^+, \mathfrak{d}^-$  on  $\partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon$  by

$$\mathfrak{d}^+(\mathbf{z}) = \lim_{t \rightarrow 0^+} p_\epsilon(\mathbf{z} + t\mathbf{n}(\mathbf{z})), \quad \mathfrak{d}^-(\mathbf{z}) = \lim_{t \rightarrow 0^+} p_\epsilon(\mathbf{z} - t\mathbf{n}(\mathbf{z})).$$

Let  $\mathfrak{d} : \partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon \rightarrow \mathbb{R}$  be given by  $\mathfrak{d} = \mathfrak{d}^+ - \mathfrak{d}^-$ , so that  $\mathfrak{d}(\mathbf{z})$  represents the discontinuity of  $p_\epsilon$  at  $\mathbf{z}$ . The next assertion provides an estimate of  $\mathfrak{d}$ .

**Assertion 7.A.** *There exist finite constants  $C_1, c_2 > 0$ , such that*

$$[\mathfrak{d}(\mathbf{z})]^2 e^{-U(\mathbf{z})/\epsilon} \leq C_1 e^{-H/\epsilon} \epsilon^{c_2 K^2}$$

for all  $\mathbf{z} \in \partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon$ .

*Proof.* Fix  $\mathbf{z} \in \partial\mathcal{X}_\epsilon$ , so that  $|\mathfrak{d}(\mathbf{z})| \leq 1$ , and  $U(\mathbf{z}) = H + (1/4)\lambda_1\delta^2$ . In this case Assertion 7.A follows from the definition of  $\delta = K\sqrt{\epsilon \log(1/\epsilon)}$ .

Fix  $\mathbf{z} \in \partial\mathcal{B}_\epsilon \setminus \partial\mathcal{X}_\epsilon$  so that, by Lemma 5.1,  $\mathbf{z} \in \partial_+\mathcal{C}_\epsilon \cup \partial_-\mathcal{C}_\epsilon$ . The proof in this case is similar to the one of Lemma 4.7 in [17]. Assume that  $\mathbf{z} \in \partial_+\mathcal{C}_\epsilon$ , the proof for  $\mathbf{z} \in \partial_-\mathcal{C}_\epsilon$  being similar. For  $\mathbf{z} \in \partial_+\mathcal{C}_\epsilon$ ,  $\mathfrak{d}^+(\mathbf{z}) = 1$  and

$$\mathfrak{d}^-(\mathbf{z}) = \frac{1}{C_\epsilon} \int_{-\infty}^{\mathbf{z} \cdot \mathbf{v}} \exp\left\{-\frac{\alpha}{2\epsilon} t^2\right\} dt,$$

so that

$$\mathfrak{d}(\mathbf{z}) = \frac{1}{C_\epsilon} \int_{\mathbf{z} \cdot \mathbf{v}}^{\infty} \exp\left\{-\frac{\alpha}{2\epsilon} t^2\right\} dt = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{\alpha}{\epsilon}}(\mathbf{z} \cdot \mathbf{v})}^{\infty} e^{-t^2/2} dt. \quad (7.1)$$

We claim that there exists a constant  $c > 0$  such that, for every  $\mathbf{z} \in \partial_+\mathcal{C}_\epsilon$ , either  $\mathbf{z} \cdot \mathbf{v} \geq c\delta$  or  $\mathbf{z} \cdot \mathbb{L}\mathbf{z} \geq c\delta^2$ . Indeed, by Lemma 7.1 and since  $v_1 > 0$ , there exists  $c > 0$  such that

$$(\lambda_1 + c) \sum_{k=2}^d \frac{v_k^2}{\lambda_k} < (v_1 - c)^2. \quad (7.2)$$

The claim is in force with this constant  $c$ . Assume it is not. This means that there exists  $\mathbf{z} \in \partial_+\mathcal{C}_\epsilon$  such that  $\mathbf{z} \cdot \mathbf{v} < c\delta$  and  $\mathbf{z} \cdot \mathbb{L}\mathbf{z} < c\delta^2$ . Since  $\mathbf{z} \in \partial_+\mathcal{C}_\epsilon$ ,  $\mathbf{z}$  can be expressed as

$$\mathbf{z} = \delta \left( \mathbf{e}_1 + \sum_{k=2}^d z_k \mathbf{e}_k \right).$$

Since  $\mathbf{v} = \sum_{1 \leq i \leq d} v_i \mathbf{e}_i$ , the condition  $\mathbf{z} \cdot \mathbf{v} < c\delta$  is equivalent to

$$v_1 - c < - \sum_{k=2}^d z_k v_k.$$

On the other hand, the condition  $\mathbf{z} \cdot \mathbb{L}\mathbf{z} < c\delta^2$  can be rewritten as

$$\sum_{k=2}^d z_k^2 \lambda_k < \lambda_1 + c.$$

Inserting the two previous bounds in (7.2) we obtain that

$$\sum_{k=2}^d \frac{v_k^2}{\lambda_k} \sum_{k=2}^d z_k^2 \lambda_k < \left( \sum_{k=2}^d z_k v_k \right)^2,$$

which contradicts to the Cauchy-Schwarz inequality. This proves the claim.

We are now in a position to prove Assertion 7.A for  $\mathbf{z} \in \partial_+ \mathcal{C}_\epsilon$ . Suppose first that  $\mathbf{z} \cdot \mathbf{v} \geq c\delta$ . Since  $\int_x^\infty \exp\{-t^2/2\} dt \leq (1/x) \exp\{-x^2/2\}$  for  $x > 0$ , by (7.1) and since  $\mathbf{z} \cdot \mathbf{v} \geq c\delta$ ,

$$0 \leq \mathfrak{d}(\mathbf{z}) \leq \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\epsilon}}{\sqrt{\alpha}(\mathbf{z} \cdot \mathbf{v})} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} \leq C \frac{\sqrt{\epsilon}}{\delta} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2}$$

for some finite constant  $C$ . On the other hand, by the Taylor expansion,

$$U(\mathbf{z}) = H + \frac{1}{2} \mathbf{z} \cdot \mathbb{L}\mathbf{z} + O(\delta^3). \quad (7.3)$$

In view of the two previous displayed equations,

$$\mathfrak{d}(\mathbf{z})^2 e^{-U(\mathbf{z})/\epsilon} \leq C e^{-H/\epsilon} \exp \left\{ -\frac{1}{2\epsilon} \mathbf{z} \cdot [\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z} \right\}.$$

By Lemma 7.2,  $\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger \geq r_0 I$ , where  $r_0 > 0$  is the smallest eigenvalue of the positive-definite matrix  $\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger$ . Hence, as  $z_1 = \delta$ ,

$$\mathbf{z} \cdot [\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z} \geq r_0 |\mathbf{z}|^2 \geq r_0 \delta^2.$$

In view of the previous two displayed equations, to complete the proof of Assertion 7.A, it remains to recall the definition of  $\delta$ .

Assume now that  $\mathbf{z}$  is such that  $\mathbf{z} \cdot \mathbb{L}\mathbf{z} \geq c\delta^2$ . In this case, Assertion 7.A is direct consequence from the bound  $|\mathfrak{d}(\mathbf{z})| \leq 1$  and from (7.3).  $\square$

The next result expresses the difference  $\nabla p_\epsilon^{(\eta)} - \mathbf{q}_\epsilon^{(\eta)}$  in terms of the function  $\mathfrak{d}$ .

**Assertion 7.B.** *For any  $\mathbf{z} \in \mathbb{R}^d$ ,*

$$\nabla p_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) = \oint_{\partial \mathcal{X}_\epsilon \cup \partial \mathcal{B}_\epsilon} \mathfrak{d}(\mathbf{y}) \phi_\eta(\mathbf{z} - \mathbf{y}) \mathbf{n}(\mathbf{y}) \sigma(d\mathbf{y}).$$

*Proof.* We first note that

$$\nabla p_\epsilon^{(\eta)}(\mathbf{z}) = \int_{\mathbb{R}^d} p_\epsilon(\mathbf{y}) (\nabla \phi_\eta)(\mathbf{z} - \mathbf{y}) d\mathbf{y}.$$

Since  $p_\epsilon$  is smooth on each domain  $\mathcal{B}_\epsilon$ ,  $\mathcal{W}_1^\epsilon$ ,  $\mathcal{W}_2^\epsilon$  and  $\mathbb{R}^d \setminus (\mathcal{B}_\epsilon \cup \mathcal{W}_1^\epsilon \cup \mathcal{W}_2^\epsilon)$ , we decompose the last integral into four integrals in these domains, and then apply divergence theorem for each integrals. For instance,

$$\begin{aligned} & \int_{\mathcal{B}_\epsilon} p_\epsilon(\mathbf{y}) (\nabla \phi_\eta)(\mathbf{z} - \mathbf{y}) d\mathbf{y} \\ &= \int_{\mathcal{B}_\epsilon} \mathbf{q}_\epsilon(\mathbf{y}) \phi_\eta(\mathbf{z} - \mathbf{y}) d\mathbf{y} + \oint_{\partial \mathcal{B}_\epsilon} \mathfrak{d}^+(\mathbf{y}) \phi_\eta(\mathbf{z} - \mathbf{y}) \mathbf{n}(\mathbf{y}) \sigma(d\mathbf{y}). \end{aligned}$$

The proof is completed by adding four identities obtained in this manner.  $\square$

*Proof of Lemma 7.5.* There exists a finite constant  $C$  such that  $\mathbb{M}\mathbb{S}^{-1}\mathbb{M}^\dagger < C\mathbb{I}$ , where  $\mathbb{I}$  stands for the  $d \times d$  identity matrix. Therefore,

$$\begin{aligned} & \|\Phi_{p_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon^{(\eta)}}\|^2 \\ &= \frac{\epsilon}{Z_\epsilon} \int_{\mathbb{R}^d} e^{-U(\mathbf{z})/\epsilon} \left[ \nabla p_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) \right] \cdot \mathbb{M}\mathbb{S}^{-1}\mathbb{M}^\dagger \left[ \nabla p_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) \right] d\mathbf{z} \\ &\leq \frac{C\epsilon}{Z_\epsilon} \int_{\mathbb{R}^d} e^{-U(\mathbf{z})/\epsilon} |\nabla p_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon^{(\eta)}(\mathbf{z})|^2 d\mathbf{z}. \end{aligned}$$

By Assertion 7.B, this expression is equal to

$$\frac{C\epsilon}{Z_\epsilon} \int_{\mathbb{R}^d} e^{-U(\mathbf{z})/\epsilon} \left| \oint_{\partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon} \mathfrak{d}(\mathbf{y}) \phi_\eta(\mathbf{z} - \mathbf{y}) \mathbf{n}(\mathbf{y}) \sigma(d\mathbf{y}) \right|^2 d\mathbf{z}.$$

Since the surface volume of  $\partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon$  is  $M[1 + o_\epsilon(1)]$ , where  $M$  is the surface volume of  $\partial\mathcal{W}_1 \cup \partial\mathcal{W}_2$ , by the Cauchy-Schwarz inequality, the last expression is bounded by

$$\frac{C\epsilon}{Z_\epsilon} \int_{\mathbb{R}^d} \oint_{\partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon} e^{-U(\mathbf{z})/\epsilon} \mathfrak{d}(\mathbf{y})^2 \phi_\eta(\mathbf{z} - \mathbf{y})^2 \sigma(d\mathbf{y}) d\mathbf{z}$$

for some finite constant  $C$ , whose value may change from line to line.

Since  $U$  is Lipschitz continuous on the compact set

$$\{\mathbf{z} : |\mathbf{z} - \mathbf{y}| \leq \eta \text{ for some } \mathbf{y} \in \partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon\},$$

there exists a finite constant  $C$ , independent of  $\epsilon$ , such that  $U(\mathbf{z}) \geq U(\mathbf{y}) - C\eta$  for  $\mathbf{y} \in \partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon$ ,  $|\mathbf{z} - \mathbf{y}| \leq \eta$ . As  $\phi_\eta(\mathbf{z} - \mathbf{y}) = 0$  if  $|\mathbf{z} - \mathbf{y}| \geq \eta$ , and as

$$\int_{\mathbb{R}^d} \phi_\eta^2(\mathbf{z}) d\mathbf{z} = \frac{C}{\eta^d},$$

the last integral is bounded by

$$\frac{C\epsilon}{Z_\epsilon} \frac{e^{C\eta/\epsilon}}{\eta^d} \oint_{\partial\mathcal{X}_\epsilon \cup \partial\mathcal{B}_\epsilon} e^{-U(\mathbf{y})/\epsilon} \mathfrak{d}(\mathbf{y})^2 \sigma(d\mathbf{y}).$$

To complete the proof of the lemma it remains to recall Assertion 7.A. □

**Lemma 7.6.** *Assume that  $\eta \ll \delta$ . There exists a finite constant  $C_1$ , independent of  $\epsilon$  and  $\eta$ , such that*

$$\|\Theta_{\mathbf{q}_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon}\|^2 \leq o_\epsilon(1) \frac{1}{Z_\epsilon} e^{-H/\epsilon} \epsilon^{d/2} \frac{\eta}{\epsilon} \left(1 + \frac{\eta}{\epsilon}\right) \left(1 + e^{C_1\eta\delta/\epsilon}\right).$$

Denote by  $\partial^\eta\mathcal{B}_\epsilon$  the neighborhood of the boundary  $\partial\mathcal{B}_\epsilon$  defined by

$$\partial^\eta\mathcal{B}_\epsilon = \{\mathbf{z} : |\mathbf{z} - \mathbf{y}| \leq \eta \text{ for some } \mathbf{y} \in \partial\mathcal{B}_\epsilon\},$$

and let  $\mathcal{B}_\epsilon^\eta = \mathcal{B}_\epsilon \setminus \partial^\eta\mathcal{B}_\epsilon$ . Since  $\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) = \mathbf{q}_\epsilon(\mathbf{z}) = 0$  if  $\mathbf{z} \notin \mathcal{B}_\epsilon^\eta \cup \partial^\eta\mathcal{B}_\epsilon$ ,

$$\begin{aligned} & \|\Theta_{\mathbf{q}_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon}\|^2 \\ &= \frac{\epsilon}{Z_\epsilon} \int_{\mathcal{B}_\epsilon^\eta \cup \partial^\eta\mathcal{B}_\epsilon} e^{-U(\mathbf{z})/\epsilon} [\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})] \cdot \mathbb{M}\mathbb{S}^{-1}\mathbb{M}^\dagger [\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})] d\mathbf{z}. \end{aligned}$$

In particular, since  $\mathbb{M}\mathbb{S}^{-1}\mathbb{M}^\dagger < C\mathbb{I}$  for some finite constant,

$$\|\Theta_{\mathbf{q}_\epsilon^{(\eta)}} - \Theta_{\mathbf{q}_\epsilon}\|^2 \leq \frac{C\epsilon}{Z_\epsilon} \int_{\mathcal{B}_\epsilon^\eta \cup \partial^\eta\mathcal{B}_\epsilon} e^{-U(\mathbf{z})/\epsilon} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|^2 d\mathbf{z}.$$

In Assertions 7.C and 7.D below, we estimate the last integral on  $\mathcal{B}_\epsilon^\eta$  and  $\partial^\eta \mathcal{B}_\epsilon$ , respectively. Lemma 7.6 follows from these two assertions.

**Assertion 7.C.** *Assume that  $\eta \ll \delta$ . There exist finite constants  $C_1, C_2$ , independent of  $\epsilon$  and  $\eta$ , such that*

$$\int_{\mathcal{B}_\epsilon^\eta} e^{-U(\mathbf{z})/\epsilon} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|^2 d\mathbf{z} \leq \frac{C_2 \delta^2}{\epsilon^3} \epsilon^{d/2} \eta^2 e^{-H/\epsilon} e^{C_1 \eta \delta / \epsilon}.$$

*Proof.* We first derive a pointwise estimate of  $|\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|$  for  $\mathbf{z} \in \mathcal{B}_\epsilon^\eta$ . Denote by  $B(\mathbf{x}; \eta)$  the ball of radius  $\eta$  centered at  $\mathbf{x} \in \mathbb{R}^d$ , namely,  $B(\mathbf{x}; \eta) = \{\mathbf{z} : |\mathbf{z} - \mathbf{x}| \leq \eta\}$ . Note that  $B(\mathbf{z}; \eta) \subset \mathcal{B}_\epsilon$  for all  $\mathbf{z} \in \mathcal{B}_\epsilon^\eta$ . Hence, by the Cauchy-Schwarz inequality and the mean value theorem,

$$\begin{aligned} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|^2 &= \left| \int_{\mathbb{R}^d} \phi_\eta(\mathbf{y}) [\nabla p_\epsilon(\mathbf{z} + \mathbf{y}) - \nabla p_\epsilon(\mathbf{z})] d\mathbf{y} \right|^2 \\ &\leq \int_{\mathbb{R}^d} \phi_\eta(\mathbf{y}) |\nabla p_\epsilon(\mathbf{z} + \mathbf{y}) - \nabla p_\epsilon(\mathbf{z})|^2 d\mathbf{y} \\ &\leq \int_{\mathbb{R}^d} \phi_\eta(\mathbf{y}) |\mathbf{y}|^2 \sum_{i,j=1}^d \sup_{\mathbf{x} \in B(\mathbf{z}; \eta)} [\nabla_{x_i, x_j}^2 p_\epsilon(\mathbf{x})]^2 d\mathbf{y}. \end{aligned} \quad (7.4)$$

By a direct computation,

$$\nabla_{x_i, x_j}^2 p_\epsilon(\mathbf{x}) = -\frac{\alpha}{C_\epsilon \epsilon} (\mathbf{x} \cdot \mathbf{v}) e^{-(\alpha/2\epsilon)(\mathbf{x} \cdot \mathbf{v})^2} v_i v_j.$$

Since  $\eta \ll \delta$  and  $\mathbf{z} \in \mathcal{B}_\epsilon^\eta$ , there exists a finite constant  $C_1$  such that  $(\mathbf{x} \cdot \mathbf{v})^2 \geq (\mathbf{z} \cdot \mathbf{v})^2 - C_1 \eta \delta$  for all  $\mathbf{x} \in B(\mathbf{z}; \eta)$ . Hence, as  $C_\epsilon = \sqrt{2\pi\epsilon/\alpha}$  and  $(\mathbf{x} \cdot \mathbf{v})^2 \leq C_2 \delta^2$ , there exists a finite constant  $C$ , independent of  $\epsilon$  and  $\eta$  such that

$$[\nabla_{x_i, x_j}^2 p_\epsilon(\mathbf{x})]^2 \leq \frac{C_2 \delta^2}{\epsilon^3} e^{-(\alpha/\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} e^{C_1 \delta \eta / \epsilon}$$

for all  $\mathbf{x} \in B(\mathbf{z}; \eta)$ .

Therefore, in view of (7.4),

$$\begin{aligned} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|^2 &\leq \frac{C_2 \delta^2}{\epsilon^3} e^{-(\alpha/\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} e^{C_1 \delta \eta / \epsilon} \int_{\mathbb{R}^d} \phi_\eta(\mathbf{y}) |\mathbf{y}|^2 d\mathbf{y} \\ &= \frac{C_2 \delta^2}{\epsilon^3} e^{-(\alpha/\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} e^{C_1 \delta \eta / \epsilon} \eta^2. \end{aligned}$$

In consequence,

$$\int_{\mathcal{B}_\epsilon^\eta} e^{-U(\mathbf{z})/\epsilon} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|^2 d\mathbf{z} \leq \frac{C_2 \delta^2}{\epsilon^3} e^{C_1 \delta \eta / \epsilon} \eta^2 \int_{\mathcal{B}_\epsilon^\eta} e^{-U(\mathbf{z})/\epsilon} e^{-(\alpha/\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} d\mathbf{z}.$$

By the Taylor expansion,  $U(\mathbf{z}) = H + (1/2) \mathbf{z} \cdot \mathbb{L} \mathbf{z} + O(\delta^3)$  for  $\mathbf{z} \in \mathcal{B}_\epsilon^\eta$ . Hence, by Lemma 7.2, the last integral is bounded by

$$C_3 e^{-H/\epsilon} \int_{\mathbb{R}^d} \exp \left\{ -\frac{1}{2\epsilon} \mathbf{z} \cdot [\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z} \right\} d\mathbf{z} = C_3 e^{-H/\epsilon} \epsilon^{d/2} \sqrt{-\det \mathbb{L}}$$

for some finite constant  $C_3$ , independent on  $\epsilon$  and  $\eta$ . This completes the proof.  $\square$

**Assertion 7.D.** Assume that  $\eta \ll \delta$ . There exist finite constants  $C_1, C_2$ , independent of  $\epsilon, \eta$ , such that

$$\int_{\partial^n \mathcal{B}_\epsilon} e^{-U(\mathbf{z})/\epsilon} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|^2 d\mathbf{z} \leq C_1 \frac{\eta \delta^{d-1}}{\epsilon} e^{-H/\epsilon} \left(1 + e^{C_2 \eta \delta / \epsilon}\right).$$

*Proof.* We first derive pointwise bounds for  $|\mathbf{q}_\epsilon(\mathbf{z})|$  and  $|\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z})|$ . For the former, we have the trivial bound

$$|\mathbf{q}_\epsilon(\mathbf{z})| \leq \frac{C_1}{C_\epsilon} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} \quad (7.5)$$

for some finite constant  $C_1$ . For the latter, by definition, by the previous inequality and by the bound on  $(\mathbf{x} \cdot \mathbf{v})^2$  in terms of  $(\mathbf{z} \cdot \mathbf{v})^2$ , obtained in the proof of the previous assertion,

$$\begin{aligned} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z})| &\leq \int_{\mathbb{R}^d} |\mathbf{q}_\epsilon(\mathbf{z} + \mathbf{y})| \phi_\eta(\mathbf{y}) d\mathbf{y} \leq \int_{\mathbb{R}^d} \frac{C_1}{C_\epsilon} e^{-(\alpha/2\epsilon)((\mathbf{z} + \mathbf{y}) \cdot \mathbf{v})^2} \phi_\eta(\mathbf{y}) d\mathbf{y} \\ &\leq \int_{\mathbb{R}^d} \frac{C_1}{C_\epsilon} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} e^{C_2 \eta \delta / \epsilon} \phi_\eta(\mathbf{y}) d\mathbf{y} \leq \frac{C_1}{C_\epsilon} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} e^{C_2 \eta \delta / \epsilon} \end{aligned}$$

for some finite constant  $C_2$ .

By the two previous estimates of  $|\mathbf{q}_\epsilon(\mathbf{z})|$  and  $|\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z})|$ , and by the Taylor expansion,

$$\begin{aligned} e^{-U(\mathbf{z})/\epsilon} |\mathbf{q}_\epsilon^{(\eta)}(\mathbf{z}) - \mathbf{q}_\epsilon(\mathbf{z})|^2 &\leq \frac{C_1}{\epsilon} e^{-H/\epsilon} e^{-(1/2\epsilon)\mathbf{z} \cdot [\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z}} \left(1 + e^{C_2 \eta \delta / \epsilon}\right) \\ &\leq \frac{C_1}{\epsilon} e^{-H/\epsilon} \left(1 + e^{C_2 \eta \delta / \epsilon}\right). \end{aligned}$$

This pointwise estimate completes the proof of the assertion since the volume of  $\partial^n \mathcal{B}_\epsilon$  is bounded by  $C_1 \eta \delta^{d-1}$ .  $\square$

**C. Proof of Lemma 5.5.** Since  $\mathbf{q}_\epsilon$  vanishes everywhere, but on the set  $\mathcal{B}_\epsilon$ ,

$$\langle \Theta_{\mathbf{q}_\epsilon}^*, \Psi_{h_{\mathcal{V}_1, \mathcal{V}_2}} \rangle = \int_{\mathcal{B}_\epsilon} \nabla h_{\mathcal{V}_1, \mathcal{V}_2}(\mathbf{z}) \cdot \Theta_{\mathbf{q}_\epsilon}^* d\mathbf{z}.$$

Note that the inner product of vector fields has been defined only for smooth vector fields, but it can be extended to weakly differentiable vector fields.

By the divergence theorem, the last integral is equal to

$$-\frac{1}{Z_\epsilon} \int_{\mathcal{B}_\epsilon} h_{\mathcal{V}_1, \mathcal{V}_2}(\mathbf{z}) e^{-U(\mathbf{z})/\epsilon} [\mathcal{L}_\epsilon p_\epsilon](\mathbf{z}) d\mathbf{z} + \int_{\partial \mathcal{B}_\epsilon} h_{\mathcal{V}_1, \mathcal{V}_2}(\mathbf{z}) [\Theta_{\mathbf{q}_\epsilon}^* \cdot \mathbf{n}(\mathbf{z})] \sigma(d\mathbf{z}), \quad (7.6)$$

where, we recall,  $\mathbf{n}(\mathbf{z})$  stands for the outer normal vector to  $\mathcal{B}_\epsilon$  at  $\mathbf{z}$ . The next lemma states that the first term is negligible. This result holds because  $p_\epsilon$  has been defined as an approximation in  $\mathcal{B}_\epsilon$  of the solution of the equation  $\mathcal{L}_\epsilon f = 0$  with some boundary conditions.

**Lemma 7.7.** We have that

$$\int_{\mathcal{B}_\epsilon} e^{-U(\mathbf{z})/\epsilon} |\mathcal{L}_\epsilon p_\epsilon(\mathbf{z})| d\mathbf{z} = o_\epsilon(1) T_\epsilon.$$

*Proof.* By definition of  $p_\epsilon$ ,

$$\nabla p_\epsilon(\mathbf{z}) = \frac{1}{C_\epsilon} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} \mathbf{v}, \quad \partial_{z_i, z_j}^2 p_\epsilon(\mathbf{z}) = -\frac{\alpha}{\epsilon C_\epsilon} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} (\mathbf{z} \cdot \mathbf{v}) v_i v_j.$$

By the Taylor expansion,  $\nabla U(\mathbf{z}) = \mathbb{L}\mathbf{z} + O(\delta^2)$ . Hence, since  $\mathbf{v}$  is the eigenvector of  $\mathbb{L}\mathbb{M} = \mathbb{L}^*\mathbb{M}$  associated to  $-\mu$ , the first order part of  $(\mathcal{L}_\epsilon p_\epsilon)(\mathbf{z})$  is equal to

$$-\frac{1}{C_\epsilon} \left\{ \mathbb{L}\mathbf{z} \cdot \mathbb{M}\mathbf{v} + O(\delta^2) \right\} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} = \frac{1}{C_\epsilon} \left\{ \mu(\mathbf{z} \cdot \mathbf{v}) + O(\delta^2) \right\} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2}.$$

By 5.2, the second order part of  $(\mathcal{L}_\epsilon p_\epsilon)(\mathbf{z})$  is equal to

$$-\frac{\alpha}{C_\epsilon} (\mathbf{z} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbb{M}\mathbf{v}) e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2} = -\frac{\mu}{C_\epsilon} (\mathbf{z} \cdot \mathbf{v}) e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2}.$$

Since  $C_\epsilon = O(\sqrt{\epsilon})$ , it follows from the two previous identities that there exists a finite constant  $C_1$  such that

$$|\mathcal{L}_\epsilon p_\epsilon(\mathbf{z})| \leq \frac{C_1 \delta^2}{\sqrt{\epsilon}} e^{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2}$$

for all  $\mathbf{z} \in \mathcal{B}_\epsilon$ .

As  $U(\mathbf{z}) = H + (1/2)\mathbf{z} \cdot \mathbb{L}\mathbf{z} + O(\delta^3)$ , for  $\mathbf{z} \in \mathcal{B}_\epsilon$ , by the previous estimate,

$$\int_{\mathcal{B}_\epsilon} e^{-U(\mathbf{z})/\epsilon} |\mathcal{L}_\epsilon p_\epsilon(\mathbf{z})| d\mathbf{z} \leq \frac{C_1 \delta^2}{\sqrt{\epsilon}} e^{-H/\epsilon} \int_{\mathcal{B}_\epsilon} e^{-(1/2\epsilon)\mathbf{z} \cdot (\mathbb{L} + \alpha\mathbf{v}\mathbf{v}^\dagger)\mathbf{z}} d\mathbf{z}. \quad (7.7)$$

It remains to estimate the last integral. Recall Lemma 7.3, and denote by  $\theta_1 = 0, \theta_2, \dots, \theta_d > 0$  the eigenvalues of the matrix  $\mathbb{L} + \alpha\mathbf{v}\mathbf{v}^\dagger$ , and by  $\mathbf{w}_i$ ,  $1 \leq i \leq d$ , the normal eigenvector corresponding to  $\theta_i$ . Let  $\mathcal{P}_a$ ,  $a \in \mathbb{R}$ , be the  $(d-1)$ -dimensional space given by

$$\mathcal{P}_a = a\mathbf{w}_1 + \langle \mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_d \rangle,$$

where  $\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle$  stands for the linear space generated by the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$ .

Since  $\mathcal{B}_\epsilon \subset \mathcal{C}_\epsilon$ , there exists  $M > 0$  such that

$$\mathcal{B}_\epsilon \subseteq \bigcup_{a: |a| \leq M\delta} \mathcal{P}_a.$$

Consider the last integral of (7.7). Perform the change of variable  $\mathbf{z} = \sum x_i \mathbf{w}_i$ , and extend the region of integration to  $\bigcup_{a: |a| \leq M\delta} \mathcal{P}_a$ , to obtain that

$$\begin{aligned} \int_{\mathcal{B}_\epsilon} e^{-(1/2\epsilon)\mathbf{z} \cdot (\mathbb{L} + \alpha\mathbf{v}\mathbf{v}^\dagger)\mathbf{z}} d\mathbf{z} &\leq C_1 \int_{-M\delta}^{M\delta} dx_1 \int_{\mathbb{R}^{d-1}} \exp \left\{ -\frac{1}{2\epsilon} \sum_{i=2}^d \theta_i x_i^2 \right\} dx_2 \cdots dx_d \\ &= C_1 M\delta (2\pi\epsilon)^{(d-1)/2} \prod_{i=2}^d \frac{1}{\theta_i^{1/2}}. \end{aligned}$$

Therefore, by (7.7),

$$\int_{\mathcal{B}_\epsilon} e^{-U(\mathbf{z})/\epsilon} |\mathcal{L}_\epsilon p_\epsilon(\mathbf{z})| d\mathbf{z} \leq \frac{C_1 \delta^3}{\epsilon} e^{-H/\epsilon} \epsilon^{d/2}.$$

This completes the proof of the lemma since  $\delta^3/\epsilon = o_\epsilon(1)$ .  $\square$

We turn to the second integral of (7.6).

**Lemma 7.8.** *We have that*

$$\int_{\partial \mathcal{B}_\epsilon} h_{\nu_1, \nu_2}(\mathbf{z}) [\Theta_{\mathbf{q}_\epsilon}^* \cdot \mathbf{n}(\mathbf{z})] \sigma(d\mathbf{z}) = [1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0}).$$

Decompose the boundary  $\partial\mathcal{B}_\epsilon$  in three pieces, denoted by  $\partial_+\mathcal{B}_\epsilon$ ,  $\partial_-\mathcal{B}_\epsilon$  and  $\partial_0\mathcal{B}_\epsilon$ , where  $\partial_+\mathcal{B}_\epsilon = \partial\mathcal{B}_\epsilon \cap \partial_+\mathcal{C}_\epsilon$ ,  $\partial_-\mathcal{B}_\epsilon = \partial\mathcal{B}_\epsilon \cap \partial_-\mathcal{C}_\epsilon$ , and  $\partial_0\mathcal{B}_\epsilon = \partial\mathcal{B}_\epsilon \setminus (\partial_+\mathcal{B}_\epsilon \cup \partial_-\mathcal{B}_\epsilon)$ .

We claim that the contribution to the integral of the piece corresponding to the boundary  $\partial_0\mathcal{B}_\epsilon$  is negligible. Since  $|h_{\nu_1, \nu_2}| \leq 1$ , this claim follows from the next assertion.

**Assertion 7.E.** *We have that*

$$\int_{\partial_0\mathcal{B}_\epsilon} |\Theta_{\mathbf{q}_\epsilon}^*(\mathbf{z})| \sigma(d\mathbf{z}) = o_\epsilon(1) T_\epsilon.$$

*Proof.* Since  $\partial_0\mathcal{B}_\epsilon \subset \partial\mathcal{X}_\epsilon$ ,

$$U(\mathbf{z}) = H + (\lambda_1/4) \delta^2 \text{ for all } \mathbf{z} \in \partial_0\mathcal{B}_\epsilon. \quad (7.8)$$

Therefore, since  $\exp\{-(\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2\} \leq 1$  and  $C_\epsilon = \sqrt{2\pi\epsilon/\alpha}$ , by definition of  $\delta$ ,

$$|\Theta_{\mathbf{q}_\epsilon}^*(\mathbf{z})| \leq \frac{C_1}{Z_\epsilon} e^{-H/\epsilon} \sqrt{\epsilon} e^{-(\lambda_1\delta^2/4\epsilon)} = \frac{C_1}{Z_\epsilon} e^{-H/\epsilon} \epsilon^{(1/2)+(\lambda_1 K^2/4)}$$

for some finite constant  $C_1$ . Since the surface volume of  $\partial_0\mathcal{B}_\epsilon$  is of order  $\delta^{d-1}$ , the statement of the assertion is straightforward consequence from this uniform bound on  $\Theta_{\mathbf{q}_\epsilon}^*$ .  $\square$

We turn to the boundaries  $\partial_+\mathcal{B}_\epsilon$  and  $\partial_-\mathcal{B}_\epsilon$ . To estimate the integral appearing in the statement of Lemma 7.8 on these sets, bounds on the equilibrium potential  $h_{\nu_1, \nu_2}$  are needed.

**Assertion 7.F.** *There exist a finite constant  $C_0$  and  $\epsilon_0 > 0$  such that for all  $\epsilon < \epsilon_0$ ,*

$$1 - h_{\nu_1, \nu_2}(\mathbf{y}) \leq \frac{C_0}{\epsilon^d} \exp\left\{\frac{U(\mathbf{y}) - H}{2\epsilon}\right\} \text{ for all } \mathbf{y} \in \partial_+\mathcal{B}_\epsilon$$

and

$$h_{\nu_1, \nu_2}(\mathbf{y}) \leq \frac{C_0}{\epsilon^d} \exp\left\{\frac{U(\mathbf{y}) - H}{2\epsilon}\right\} \text{ for all } \mathbf{y} \in \partial_-\mathcal{B}_\epsilon. \quad (7.9)$$

*Proof.* Consider first (7.9). If  $\mathbf{y} \in \partial_-\mathcal{B}_\epsilon$  satisfies  $U(\mathbf{y}) \geq H$ , then (7.9) is obvious for all sufficiently small  $\epsilon$ . Otherwise,  $\mathbf{y} \in \mathcal{W}_2$ , and the result follows by Proposition 6.11.

The proof of the first claim of the Assertion is analogous. The previous arguments provide an upper bound for  $h_{\nu_2, \nu_1}$  which is the function  $1 - h_{\nu_1, \nu_2}$ .  $\square$

We are now in a position to prove Lemma 7.8 at the boundaries  $\partial_+\mathcal{B}_\epsilon$  and  $\partial_-\mathcal{B}_\epsilon$ .

**Assertion 7.G.** *For sufficiently large  $K$ , we have that*

$$\begin{aligned} \int_{\partial_+\mathcal{B}_\epsilon} [1 - h_{\nu_1, \nu_2}(\mathbf{z})] [\Theta_{\mathbf{q}_\epsilon}^* \cdot \mathbf{n}(\mathbf{z})] \sigma(d\mathbf{z}) &= o_\epsilon(1) T_\epsilon, \\ \int_{\partial_-\mathcal{B}_\epsilon} h_{\nu_1, \nu_2}(\mathbf{z}) [\Theta_{\mathbf{q}_\epsilon}^* \cdot \mathbf{n}(\mathbf{z})] \sigma(d\mathbf{z}) &= o_\epsilon(1) T_\epsilon. \end{aligned}$$

*Proof.* We concentrate on the first claim, the proof of the second one being similar. By Assertion 7.F and since  $C_\epsilon = O(\sqrt{\epsilon})$ ,

$$\begin{aligned} &\left| \int_{\partial_+\mathcal{B}_\epsilon} [1 - h_{\nu_1, \nu_2}(\mathbf{z})] [\Theta_{\mathbf{q}_\epsilon}^* \cdot \mathbf{n}(\mathbf{z})] \sigma(d\mathbf{z}) \right| \\ &\leq \frac{C_1 \epsilon^{1/2-d}}{Z_\epsilon} \int_{\partial_+\mathcal{B}_\epsilon} \exp\left\{\frac{U(\mathbf{z}) - H}{2\epsilon}\right\} \exp\left\{-\frac{U(\mathbf{z})}{\epsilon} - \frac{\alpha}{2\epsilon}(\mathbf{z} \cdot \mathbf{v})^2\right\} \sigma(d\mathbf{z}) \end{aligned}$$

for some finite constant  $C_1$ . The exponential terms can be written as  $-(1/\epsilon)H - (1/2\epsilon)[U(\mathbf{z}) - H] - (\alpha/2\epsilon)(\mathbf{z} \cdot \mathbf{v})^2$ . By the Taylor expansion of  $U$ , the expression on the right hand side of the previous displayed equation is bounded by

$$\begin{aligned} & \frac{C_1 \epsilon^{1/2-d}}{Z_\epsilon} e^{-H/\epsilon} \int_{\partial_+ \mathcal{B}_\epsilon} \exp \left\{ -\frac{1}{4\epsilon} \mathbf{z} \cdot [\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z} \right\} \sigma(d\mathbf{z}) \\ & \leq \frac{C_1 \epsilon^{1/2-d}}{Z_\epsilon} e^{-H/\epsilon} \int_{\partial_+ \mathcal{B}_\epsilon} \exp \left\{ -\frac{\gamma}{4\epsilon} \|\mathbf{z}\|^2 \right\} \sigma(d\mathbf{z}), \end{aligned}$$

where  $\gamma > 0$  is the smallest eigenvalue of the positive-definite matrix  $\mathbb{L} + 2\alpha \mathbf{v} \mathbf{v}^\dagger$ . Since  $\|\mathbf{z}\|^2 \geq \delta^2$  for  $\mathbf{z} \in \partial_+ \mathcal{B}_\epsilon$ , the last integral is less than or equal to  $C_1 \epsilon^{\gamma K^2/4} \delta^{d-1}$ , which completes the proof of the assertion in view of the definition of  $T_\epsilon$  provided that  $K$  is sufficiently large.  $\square$

Next assertion completes the proof of Lemma 7.8.

**Assertion 7.H.** *We have that*

$$\int_{\partial_+ \mathcal{B}_\epsilon} \Theta_{\mathbf{q}_\epsilon}^* \cdot \mathbf{n}(\mathbf{z}) \sigma(d\mathbf{z}) = [1 + o_\epsilon(1)] T_\epsilon \omega(\mathbf{0})$$

*Proof.* Since  $\mathbf{n}(\mathbf{z}) = \mathbf{e}_1$  for  $\mathbf{z} \in \partial_+ \mathcal{B}_\epsilon$ , by the Taylor expansion, the left hand side of the previous equation can be written as

$$[1 + o_\epsilon(1)] \frac{\epsilon}{Z_\epsilon} \sqrt{\frac{\alpha}{2\pi\epsilon}} e^{-H/\epsilon} (\mathbf{e}_1 \cdot \mathbb{M} \mathbf{v}) \int_{\partial_+ \mathcal{B}_\epsilon} e^{-(1/2\epsilon) \mathbf{z} \cdot [\mathbb{L} + \alpha \mathbf{v} \mathbf{v}^\dagger] \mathbf{z}} \sigma(d\mathbf{z}). \quad (7.10)$$

Let  $\theta_k = (v_k \lambda_1)/(v_1 \lambda_k)$ ,  $2 \leq k \leq d$ , and define the variable  $\mathbf{y} = (y_2, \dots, y_d)$  by

$$\mathbf{z} = \mathbf{e}_1 + \sum_{k=2}^d (y_k - \theta_k) \mathbf{e}_k,$$

An elementary computation, based on the identity provided by Lemma 7.1, yields that

$$\mathbf{z} \cdot (\mathbb{L} + \alpha \mathbf{v} \mathbf{v}^\dagger) \mathbf{z} = \mathbf{y} \cdot (\tilde{\mathbb{L}} + \alpha \mathbf{w} \mathbf{w}^\dagger) \mathbf{y},$$

where  $\mathbf{w}$  is the  $(d-1)$ -dimensional vector given by  $\mathbf{w} = (v_2, \dots, v_d)$  and  $\tilde{\mathbb{L}}$  is the  $(d-1) \times (d-1)$  diagonal matrix  $\text{diag}(\lambda_2, \dots, \lambda_d)$ .

Perform the change of variables presented in the penultimate displayed equation to write the last integral in (7.10) as

$$\int_{D_\epsilon} e^{-(1/2\epsilon) \mathbf{y} \cdot [\tilde{\mathbb{L}} + \alpha \mathbf{w} \mathbf{w}^\dagger] \mathbf{y}} d\mathbf{y},$$

where  $D_\epsilon \subset \mathbb{R}^{d-1}$  is the domain of integration obtained from  $\partial_+ \mathcal{B}_\epsilon$  by the change of variables. By Lemma 7.1 and a Taylor expansion  $U(\delta, -\delta\theta_1, \dots, -\delta\theta_d) < H + (1/4)\lambda_1\delta^2$  for all  $\epsilon$  small enough. In particular, for  $\epsilon$  small enough,  $D_\epsilon$  contains a ball centered at the origin and of radius  $r\delta$  for some  $r > 0$ ,  $D_\epsilon \supset B(\mathbf{0}, r\delta)$ . Furthermore, it is easy to verify that  $\tilde{\mathbb{L}} + \alpha \mathbf{w} \mathbf{w}^\dagger$  is positive definite and hence the last integral is equal to

$$[1 + o_\epsilon(1)] (2\pi\epsilon)^{(d-1)/2} \left\{ \det(\tilde{\mathbb{L}} + \alpha \mathbf{w} \mathbf{w}^\dagger) \right\}^{-1/2}.$$



Since  $\det(\mathbb{A} + \mathbf{x}\mathbf{y}^\dagger) = (1 + \mathbf{y}^\dagger \mathbb{A}^{-1} \mathbf{x}) \det \mathbb{A}$ , by Lemma 7.1,  $\det(\tilde{\mathbb{L}} + \alpha \mathbf{w}\mathbf{w}^\dagger)$  is equal to

$$(1 + \alpha \mathbf{w}^\dagger \tilde{\mathbb{L}}^{-1} \mathbf{w}) \det \tilde{\mathbb{L}} = \alpha \left( \frac{1}{\alpha} + \sum_{k=2}^d \frac{v_k^2}{\lambda_k} \right) \prod_{i=2}^d \lambda_i = \alpha \frac{v_1^2}{\lambda_1} \prod_{i=2}^d \lambda_i.$$

On the other hand, since  $\mathbf{v}$  is the eigenvector of  $\mathbb{L}\mathbb{M}$  associated to the eigenvalue  $-\mu$ ,

$$\mathbf{e}_1 \cdot \mathbb{M} \mathbf{v} = \mathbf{e}_1 \cdot \mathbb{L}^{-1} \mathbb{L} \mathbb{M} \mathbf{v} = -\mu \mathbf{e}_1 \cdot \mathbb{L}^{-1} \mathbf{v} = \frac{\mu}{\lambda_1} v_1.$$

To complete the proof of the assertion, it remains to recollect all estimates, and to recall the definition of  $\omega(\mathbf{0})$ , introduced in (2.17), and the one of  $T_\epsilon$ , given in (5.4).  $\square$

## 8. PROOF OF THEOREM 2.4

Recall from (2.13) that we denote by  $H_{\mathbf{x}, \mathbf{y}}$  the height of the saddle point between  $\mathbf{x}$  and  $\mathbf{y} \in \mathbb{R}^d$ .

For  $U(\mathbf{m}_1) < r < U(\boldsymbol{\sigma}_1)$ , let  $\mathcal{N}_r$  be the neighborhood of  $\mathbf{m}_1$  given by all points which are connected to  $\mathbf{m}_1$  by a continuous path whose height lies below  $r$ :

$$\mathcal{N}_r = \{ \mathbf{x} \in \mathbb{R}^d : H_{\mathbf{x}, \mathbf{m}_1} \leq r \}.$$

An elementary computation shows that there exists a finite constant  $C_0 = C_0(r)$  such that  $(\mathcal{L}_\epsilon U)(\mathbf{x}) \leq C_0$  on  $\mathcal{N}_r$ . Thus, if  $H_r$  stands for the hitting time of the boundary of  $\mathcal{N}_r$ , which is finite because, by condition (P4), the process is positive recurrent,

$$\mathbb{E}_{\mathbf{x}}[U(X_{t \wedge H_r}^\epsilon)] - U(\mathbf{x}) \leq C_0 \mathbb{E}_{\mathbf{x}}[H_r]$$

for all  $t \geq 0$ . Letting  $t \rightarrow \infty$ , we obtain that for all  $\epsilon > 0$ ,  $U(\mathbf{m}_1) < s < r < U(\boldsymbol{\sigma}_1)$ ,  $\mathbf{x} \in \mathcal{N}_s$ ,

$$\mathbb{E}_{\mathbf{x}}[H_r] \geq (r - s)/C_0. \quad (8.1)$$

Since  $\mathbf{m}_1$  is a non-degenerate critical point of  $U$ , there exists a finite constant  $C_1$  such that  $\|\nabla U(\mathbf{x})\| \leq C_1 \|\mathbf{x} - \mathbf{m}_1\|$  for all  $\mathbf{x} \in B_1(\mathbf{m}_1)$ . In particular, on  $B_{r\sqrt{\epsilon}}(\mathbf{m}_1)$ ,  $r > 0$ ,  $\|\nabla U\| \leq C_1 r \sqrt{\epsilon}$ . Hence, by (6.2),  $\nu_{B_{r\sqrt{\epsilon}}(\mathbf{m}_1)} \leq C_1 r^2 \epsilon^{-1}$ , so that

$$\nu_{B_{r\sqrt{\epsilon}}(\mathbf{m}_1)} \epsilon \leq C_0(r). \quad (8.2)$$

**Lemma 8.1.** *Let  $w(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[H_{\mathcal{V}_2}]$ . There exists a finite constant  $C_0$  such that*

$$\sup_{\mathbf{x} \in B_{\sqrt{\epsilon}}(\mathbf{m}_1)} w(\mathbf{x}) \leq C_0 \inf_{\mathbf{x} \in B_{\sqrt{\epsilon}}(\mathbf{m}_1)} w(\mathbf{x}).$$

*Proof.* Fix  $\mathbf{x}, \mathbf{x}' \in B_{\sqrt{\epsilon}}(\mathbf{m}_1)$ . Let  $G_{\mathcal{V}_2^c}$  be the Green function associated to the diffusion killed at  $\mathcal{V}_2$ . Since  $w$  solves (2.6) with  $\mathbf{g} = 1$ ,  $\mathbf{b} = 0$ , by Lemma 6.3,

$$w(\mathbf{x}) = \int_{\mathcal{V}_2^c} G_{\mathcal{V}_2^c}(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

Fix  $\mathbf{y} \in \mathcal{V}_2^c \setminus B_{2\sqrt{\epsilon}}(\mathbf{m}_1)$ . The function  $G_{\mathcal{V}_2^c}(\cdot, \mathbf{y})$  is non-negative and harmonic in  $B_{2\sqrt{\epsilon}}(\mathbf{m}_1)$ . Hence, by Lemma 6.1 and (8.2),  $G_{\mathcal{V}_2^c}(\mathbf{x}, \mathbf{y}) \leq C_0 G_{\mathcal{V}_2^c}(\mathbf{x}', \mathbf{y})$ . The right hand side of the previous formula is thus bounded above by

$$C_0 \int_{\mathcal{V}_2^c \setminus B_{2\sqrt{\epsilon}}(\mathbf{m}_1)} G_{\mathcal{V}_2^c}(\mathbf{x}', \mathbf{y}) d\mathbf{y} + \int_{B_{2\sqrt{\epsilon}}(\mathbf{m}_1)} G_{\mathcal{V}_2^c}(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

The first term is bounded by  $C_0 w(\mathbf{x}')$ , while the second one, in view of Lemma 6.4, is less than or equal to  $C_0 a_d(\epsilon)$ , where  $a_d(\epsilon) = \epsilon$ ,  $d \geq 3$ , and  $a_2(\epsilon) = \epsilon \log \epsilon^{-1}$ .

Fix  $U(\mathbf{m}_1) < r < U(\boldsymbol{\sigma}_1)$ . Since  $H_r \leq H_{\mathcal{V}_2}$ , where  $H_r$  has been introduced above (8.1),  $w(\mathbf{x}') \geq [U(\boldsymbol{\sigma}_1) - U(\mathbf{m}_1)]/C_0$ . We may therefore bound  $a_d(\epsilon)$  by  $w(\mathbf{x}')$  to complete the proof of the lemma.  $\square$

**Lemma 8.2.** *We have that*

$$\mathbb{E}_{\mathbf{m}_1}[H_{\mathcal{V}_2}] = (1 + o_\epsilon(1)) \frac{1}{\text{cap}(B_\epsilon(\mathbf{m}_1), \mathcal{V}_2)} \int_{\mathbb{R}^d} h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{y}) \mu_\epsilon(d\mathbf{y}).$$

*Proof.* Recall that  $w(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[H_{\mathcal{V}_2}]$ . Since  $w$  solves  $\mathcal{L}_\epsilon w = -1$  on  $B_{\sqrt{\epsilon}}(\mathbf{m}_1)$ , and since, by (8.2),  $\nu_{B_{\sqrt{\epsilon}}(\mathbf{m}_1)}(\sqrt{\epsilon})^2 \leq C_0$ , by Lemma 6.2 with  $R = \epsilon$ ,

$$\begin{aligned} \sup_{\mathbf{x} \in B_\epsilon(\mathbf{m}_1)} w(\mathbf{x}) &\leq w(\mathbf{m}_1) + C_0 \epsilon^{\alpha/2} \left( \text{osc}(w, B_{\sqrt{\epsilon}}(\mathbf{x})) + R_0 \right) \\ &\leq w(\mathbf{m}_1) + C_0 \epsilon^{\alpha/2} \left( \sup_{\mathbf{x}' \in B_{\sqrt{\epsilon}}(\mathbf{m}_1)} w(\mathbf{x}') + \sqrt{\epsilon} \right), \end{aligned}$$

where we used the fact that  $w$  is non-negative in the last inequality and we replaced  $R_0$  by  $\sqrt{\epsilon}$ . Recall from the proof of the previous lemma that  $w(\mathbf{m}_1) > c_0 > 0$  for some positive constant  $c_0$  independent of  $\epsilon$ . Hence, by Lemma 8.1, the previous expression is bounded by

$$w(\mathbf{m}_1) + C_0 \epsilon^{\alpha/2} \left( w(\mathbf{m}_1) + \sqrt{\epsilon} \right) \leq w(\mathbf{m}_1) + C_0 \epsilon^{\alpha/2} w(\mathbf{m}_1),$$

so that

$$\sup_{\mathbf{x} \in B_\epsilon(\mathbf{m}_1)} w(\mathbf{x}) \leq (1 + o_\epsilon(1)) w(\mathbf{m}_1).$$

A lower bound for  $\inf_{\mathbf{x} \in B_\epsilon(\mathbf{m}_1)} w(\mathbf{x})$  is derived analogously.

Recall from (3.12) the definition of the harmonic measure  $\nu_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}$ . Since it is concentrated on  $\partial B_\epsilon(\mathbf{m}_1)$ , it follows from the previous estimates that

$$w(\mathbf{m}_1) = (1 + o_\epsilon(1)) \int_{\partial B_\epsilon(\mathbf{m}_1)} w(\mathbf{y}) \nu_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}(d\mathbf{y}),$$

To complete the proof of the lemma, it remains to recall identity (3.14).  $\square$

**Lemma 8.3.** *We have that*

$$\int_{\mathbb{R}^d} h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} d\mathbf{y} = (1 + o_\epsilon(1)) \frac{(2\pi\epsilon)^{d/2} e^{-U(\mathbf{m}_1)/\epsilon}}{\sqrt{\det[(\text{Hess } U)(\mathbf{m}_1)]}}. \quad (8.3)$$

*Proof.* We estimate separately the integral on different parts. Recall from (5.1) the definition of  $\delta$ . We claim that

$$\int_{B_\delta(\mathbf{m}_1)} h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} d\mathbf{y} = (1 + o_\epsilon(1)) \frac{(2\pi\epsilon)^{d/2} e^{-U(\mathbf{m}_1)/\epsilon}}{\sqrt{\det[(\text{Hess } U)(\mathbf{m}_1)]}}. \quad (8.4)$$

Indeed, by Proposition 6.11, on  $B_\delta(\mathbf{m}_1)$ ,  $h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{y}) = 1 - h_{\mathcal{V}_2, B_\epsilon(\mathbf{m}_1)}^*(\mathbf{y}) = 1 + o_\epsilon(1)$ . A Taylor expansion of  $U$  around  $\mathbf{m}_1$  together with Gaussian estimates permits to conclude.

Let  $\kappa_1$  the smallest eigenvalue of  $(\text{Hess } U)(\mathbf{m}_1)$ . There exists  $r_0 > 0$  such that  $U(\mathbf{x}) - U(\mathbf{m}_1) \geq (1/4)\kappa_1 \|\mathbf{x}\|^2$  for all  $\mathbf{x} \in B_{r_0}(\mathbf{m}_1)$ . We claim that

$$\int_{B_{r_0}(\mathbf{m}_1) \setminus B_\delta(\mathbf{m}_1)} h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} d\mathbf{y} = o_\epsilon(1) \epsilon^{d/2} e^{-U(\mathbf{m}_1)/\epsilon}. \quad (8.5)$$

By the bound on  $U$  and since the harmonic function is bounded by 1, the integral is less than or equal to

$$e^{-U(\mathbf{m}_1)/\epsilon} \int_{B_{r_0}(\mathbf{m}_1) \setminus B_\delta(\mathbf{m}_1)} e^{-(1/4)(\kappa_1/\epsilon)\|\mathbf{y}\|^2} d\mathbf{y}.$$

A change of variables and an elementary computation yields that the integral is equal to  $o_\epsilon(1) \epsilon^{d/2}$ , which completes the proof of (8.5).

Let  $a_0 = \inf_{\mathbf{x} \in \partial B_{r_0}(\mathbf{m}_1)} U(\mathbf{x}) > U(\mathbf{m}_1)$ . Assume that  $a_0 < U(\boldsymbol{\sigma}_1)$ . If this is not the case replace  $a_0$  by  $a'_0$  where  $U(\mathbf{m}_1) < a'_0 < U(\boldsymbol{\sigma}_1)$ . Let  $\mathcal{S}(a_0) = \{\mathbf{x} \in \mathbb{R}^d : U(\mathbf{x}) \geq a_0\}$ . It follows from this bound and (2.15) that

$$\int_{\mathcal{S}(a_0)} h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} d\mathbf{y} = e^{-U(\mathbf{m}_1)/\epsilon} e^{-a/\epsilon} \quad (8.6)$$

for some  $a > 0$ , which is exponentially smaller than the right hand side of (8.3)

It remains to estimate the integral over the set  $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^d : U(\mathbf{x}) \leq a_0, U(\mathbf{x}) = U(\mathbf{z}(\mathbf{x}, \mathbf{m}_2))\}$ . By property (P1), the set  $\mathcal{A}$  is bounded. On this set, by Proposition 6.10,  $h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{x}) \leq C_0 \epsilon^{-d} \exp\{-\epsilon^{-1}[U(\boldsymbol{\sigma}_1) - U(\mathbf{x})]\}$ . Therefore,

$$\int_{\mathcal{A}} h_{B_\epsilon(\mathbf{m}_1), \mathcal{V}_2}^*(\mathbf{y}) e^{-U(\mathbf{y})/\epsilon} d\mathbf{y} \leq C_0 \epsilon^{-d} \int_{\mathcal{A}} e^{-U(\boldsymbol{\sigma}_1)/\epsilon} d\mathbf{y} \leq C_0 \epsilon^{-d} e^{-U(\mathbf{m}_1)/\epsilon} e^{-a/\epsilon}$$

for some  $a > 0$ . This completes the proof of the lemma.  $\square$

*Proof of Theorem 2.4.* It is enough to put together the estimates of Lemmata 8.2, 8.3 with the estimate of the capacity, stated in Theorem 2.3 with  $\mathcal{V}_1 = B_\epsilon(\mathbf{m}_1)$ .  $\square$

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